

# Directed Percolation with Colors and Flavors

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A model of directed percolation processes with colors and flavors that is equivalent to a population model with many species near their extinction thresholds is presented. We use renormalized field theory and demonstrate that all renormalizations needed for the calculation of the universal scaling behavior near the multicritical point can be gained from the one-species Gribov process (Reggeon field theory). In addition this universal model shows an instability that generically leads to a total asymmetry between each pair of species of a cooperative society, and finally to unidirectionality of the interspecies couplings. It is shown that in general the universal multicritical properties of unidirectionally coupled directed percolation processes with linear coupling can also be described by the model. Consequently the crossover exponent describing the scaling of the linear coupling parameters is given by  $\Phi=1$  to all orders of the perturbation expansion. As an example of unidirectionally coupled directed percolation, we discuss the population dynamics of the tournaments of three species with colors of equal flavor.

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**KEY WORDS:** Multicolored directed percolation; field-theoretic renormalization group; stochastic population dynamics.

## I. INTRODUCTION

Nonequilibrium processes, their stationary states and their phase transitions have been of considerable interest in natural science as well as in medicine and sociology for many years. Here we are interested in processes that can be modelled by growth and decay of populations with spatially local interaction rules. The transition between survival and extinction of a population is a nonequilibrium continuous phase transition phenomenon and is characterized by universal scaling laws. It is well known that also for

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systems far from equilibrium the concept of universality classes with respect to their critical properties in the vicinity of a continuous phase transition is applicable. For the description of transitions in systems that show active and absorbing inactive states, percolation models play an outstanding role. Some years ago it was conjectured<sup>(1,2)</sup> that Markovian growth models with one-component order parameters displaying a transition into an absorbing state in the absence of any special conservation law generically belong to the universality class of directed percolation (DP). Besides DP<sup>(3-5)</sup> this universality class includes e.g., Reggeon field theory (RFT),<sup>(6-8)</sup> the contact process,<sup>(9-11)</sup> certain cellular automata<sup>(12)</sup> and some catalysis models<sup>(13,14)</sup> (for a recent review of DP processes see ref. 15).

Despite the fact that a large variety of different models belong to the DP universality class, *there is still no experiment where the critical behavior of DP was seen.*<sup>(16)</sup> In a recent paper,<sup>(17)</sup> Hinrichsen compares suggested experiments and discusses possible reasons why the observation of DP critical exponents is obscured or even impossible. One of these reasons might be that the basic feature of the DP class, the existence of an absorbing state, is quite difficult to realize in nature. Small fluctuations will always affect this state and may be strong enough to soften the transition like a small particle source, which works as an external field.<sup>(18)</sup> Another reason might be the influence of spatial quenched disorder which is abundant in reality. We have shown<sup>(19)</sup> that in contrast to equilibrium systems, the critical scaling properties of DP processes are not only altered by frozen randomness, but fully destroyed.

For the analytic description of universal behavior near a critical nonequilibrium transition, it is often useful to model the universality class by mesoscopic stochastic processes involving the order parameter and other relevant fields. In case of the DP class a representation by the Langevin equation for the time-development of the particle density, the Gribov process (the stochastic version of the so-called Schlögl model<sup>(20)</sup>), is appropriate.<sup>(1)</sup> The name Gribov process was coined by Grassberger who showed that RFT is a Markov process in disguise rather than a quantum theory.<sup>(8)</sup> On the level of a formulation of stochastic processes by means of path integrals, there is superficially no difference between RFT and the Gribov process.<sup>(1,4,8)</sup> However, RFT uses creation and annihilation operators for particles as the principal fields in contrast to the particle density and its conjugate response field used in the Gribov process. Microscopically the RFT-description of DP starts with special reactions between diffusing individuals on a given  $d$ -dimensional lattice such as birth:  $X \rightarrow X + X$ , competition:  $X + X \rightarrow X$ , and death:  $X \rightarrow 0$ . These reactions are represented by a master equation that is mapped onto a second-quantized bosonic operator representation, which is in turn mapped onto a bosonic

field theory using the continuum limit.<sup>(21, 22)</sup> At the critical point the rates of birth and death have to balance to yield a vanishing overall production rate of individuals.

In the recent years it became clear that *spontaneous single particle decay with a nonvanishing death rate is, besides the existence of an absorbing state, a further necessary condition to put multiparticle reaction-diffusion processes into a percolation universality class.* Nonvanishing autocatalytic creation processes correlate particles in local clusters embedded in the vacuum. Then, near criticality, the subsequent uncorrelated spontaneous decay produces the random percolation pattern consisting of the typical selfsimilar patchwork of a dense particle distribution as seen in so many simulation pictures (see e.g., ref. 15). *A fluctuating density description is here applicable.*<sup>(1, 23)</sup> In contrast, the replacement of the single particle decay by a two particle reaction  $X + X \rightarrow 0$  separates particles and leads to strong anticorrelations.<sup>(24)</sup> The pictures seen in corresponding simulations consist of separated lonely wanderers that only sporadically interact and no clustering of particles sets in. A naive stochastic density description for the fundamental field fails in this case. One has to resort to the bosonic creation-annihilation operator formulation.<sup>(24)</sup> As Cardy and Täuber<sup>(25)</sup> showed: a nonvanishing birth rate leads then to compact growth well described by mean field theory. Only for spatial dimensions  $d \leq 2$ , where Brownian paths have a probability of return equal to one, the two-particle annihilation process is efficient enough to produce an effective one-particle decay rate from combined branching and annihilation. One finds then indeed a DP transition from an absorbing to an active phase at a non-zero positive birth rate. A formal proof of the equivalence of the density description with the bosonic creation-annihilation operator description is given in Appendix D.

When studying processes belonging to the DP universality class, one considers usually only one species of particles. In mathematical biology<sup>(26, 27)</sup> on the other hand, processes which comprise several distinct species (henceforth labelled by different colors) are of prime interest. Also, models of this type are of relevance for surface growth.<sup>(28)</sup> In this paper we study such *multicolored directed percolation processes* (MDP) by means of the Gribov process for several species. A brief account of this work has been presented in ref. 29. The paper presents a renormalized field theory of this process in detail. In our approach the species are grouped into several flavor classes, so that all species in a flavor class have the same diffusion constants and intraspecies interactions. It results a symmetry under permutations of the species within a flavor class. One of the key results of our work is the instability of this color-symmetry if interspecies interactions are introduced.

The plan of presentation is the following: In the next chapter we introduce the model and define its renormalization and the one-loop calculation in the third chapter. In Chapter IV, we present the general renormalization group analysis and find the asymptotic scaling behavior to one-loop order. In Chapter V, we include the two-loop results of the appendices and show the crossover of the couplings between the different species to unidirectionality. In Chapter VI, we present our considerations on symmetries and the general fixed point properties of the model. We show that the permutation symmetry of a multicolored process is spontaneously broken. In Chapter VII, we show that the universal multicritical features of a recently introduced model of unidirectional coupled directed percolation processes (UCDP) by Täuber *et al.*,<sup>(30)</sup> which contains an additional linear coupling between the species, is completely described by the MDP class. In Chapter IX we show that the asymptotic unidirectionality of MDP can be described in a concise fashion by tournaments. We give an application to biomathematics. In an epilogue we resume the results and give an outlook. Four appendices present technical details, e.g., the  $\varepsilon$ -expansion of the DP-exponents, known for a long time, but as yet unpublished.

## II. THE MODEL

The mesoscopic description of the dynamics of physical systems is based on a correct choice of the complete set of fundamental slowly-developing fields. In general these are the order parameter densities and the densities of conserved quantities. As we have argued in the introduction, the multispecies processes under consideration are completely described by the particle densities  $n(\mathbf{x}, t) = (n_1(\mathbf{x}, t), n_2(\mathbf{x}, t), \dots)$  of the percolating colored and flavored individuals. We assume that there does not exist any conservation law.

Next one has to find out the general form of the stochastic equations of motions of the fundamental fields as timelocal (Markovian) Langevin equations. These Langevin equations have to respect symmetries and general principles characterizing the universality class under consideration. The MDP-class is characterized by the following four principles:

1. Errorfree self-reproduction (“birth”) and spontaneous annihilation (“death”) of individuals. The rates for birth and death may be different for each species.
2. Interaction between the individuals (“competition,” “saturation”) with color-dependent couplings.

3. Diffusion (“motion,” “spreading”) of the individuals in a  $d$ -dimensional space with flavor-depending transport coefficients.

4. The states with at least one extinct species are absorbing.

In the language of chemistry, the MDP may be realized microscopically by an autocatalytic reaction scheme of the form  $X_\alpha \leftrightarrow 2X_\alpha$ ,  $X_\alpha \rightarrow 0$ ,  $X_\alpha + X_\beta \rightarrow kX_\alpha + lX_\beta$ , where the last reaction subsumes the interactions of individuals with colors  $\alpha$ ,  $\beta$ , and  $k$ ,  $l$  may be the integers 0, 1. A description in terms of particle densities typically arises from a coarse-graining procedure where a large number of microscopic degrees of freedom are averaged out. The influence of these is simply modelled by Gaussian noise-terms in the Langevin equation which however have to respect the absorbing state condition. The stochastic reaction-diffusion equations for the particle densities in accordance with the four principles given above are of the form

$$\partial_t n_\alpha(\mathbf{x}, t) = \lambda_\alpha \nabla^2 n_\alpha(\mathbf{x}, t) + R_\alpha(n(\mathbf{x}, t)) n_\alpha(\mathbf{x}, t) + \zeta_\alpha(\mathbf{x}, t) \quad (1)$$

where the first term on the right hand side models the (diffusive) motion, and the  $R_\alpha$  are the overall reproduction rates of the particles with color  $\alpha$ . These deterministic terms are constructed proportional to  $n_\alpha$  in order to ensure the existence of an absorbing state for each species. Near the absorbing transition the particle densities  $n$  are small quantities. Expanding the rates  $R_\alpha$  in powers of  $n$  results in

$$R_\alpha(n) = -\lambda_\alpha \left( \tau_\alpha + \frac{1}{2} \sum_\beta g_{\alpha\beta} n_\beta + \dots \right) \quad (2)$$

The Gaussian noises  $\zeta_\alpha(\mathbf{r}, t)$  must also respect the absorbing state condition, whence

$$\begin{aligned} \langle \zeta_\alpha(\mathbf{x}, t) \zeta_\beta(\mathbf{x}', t') \rangle &= 2D_{\alpha\beta}(n(\mathbf{x}, t)) \delta(\mathbf{x} - \mathbf{x}') \delta(t - t') + \dots \\ &= \lambda_\alpha g_{\alpha\beta} \delta_{\alpha,\beta} n_\alpha(\mathbf{x}, t) \delta(\mathbf{x} - \mathbf{x}') \delta(t - t') + \dots \end{aligned} \quad (3)$$

Subleading terms in the expansions (2,3) as well as additional terms with derivatives of the spatial  $\delta$ -function in the first line of Eq. (3) are not displayed. It can be shown that they are irrelevant in the renormalization group sense as long as the stability condition  $\sum_{\alpha,\beta} \lambda_\alpha g_{\alpha\beta} n_\alpha n_\beta \geq 0$  for all  $n_\alpha \geq 0$  is fulfilled. The breakdown of this condition signals the occurrence of a discontinuous transition to compact growth and the appearance of tricritical phenomena at the border between first and second order transitions. Such a behavior is expected in microscopic models based on more complicated particle reactions.<sup>(31)</sup> The “temperature” variables  $\tau_\alpha$  measure

the difference of the rates of death and birth of individuals with color  $\alpha$ . Thus the temperatures may be positive or negative. We are interested in the case where all  $\tau_\alpha \approx 0$  (up to fluctuation corrections) which defines the multicritical region. Under these conditions all the species live on the border of extinction.

Species with equal kinetic coefficients  $\lambda_\alpha$  and intraspecies coupling constants  $g_\alpha, g_{\alpha\alpha}$  belong to the same flavor class. Note that we do not demand the equality of the temperatures  $\tau_\alpha$ . The equations of motion (2,3) exhibit color-symmetry at the multicritical point with all  $\tau_\alpha=0$ , if all the other parameters depend only on the flavors of the colors.

The next step is a mean field investigation of homogeneous steady state solutions of the equations of motion. Neglecting all fluctuations in Eq. (1) and using the expansion (2) up to first order in  $n$ , we find easily that all  $M_\alpha := \langle n_\alpha(\mathbf{x}, t) \rangle_{\text{steady state}} = 0$  as long as all  $\tau_\alpha > 0$ , meaning the vacuum is absorbing for each color. As soon as some of the temperatures become negative, stationary solutions with  $M_\alpha > 0$  emerge, satisfying the corresponding equations  $\sum_\beta g_{\alpha\beta} M_\beta = -2\tau_\alpha$ . Thus, in general a multitude of hypersurfaces of first and second order transitions exist in the phase space spanned by the relevant temperature variables  $\{\tau_\alpha\}$ , which separate the phases where a specific color becomes extinct. Whenever  $(\sum_\beta g_{\alpha\beta} M_\beta + 2\tau_\alpha)$  changes from a negative to a positive value, the order parameter  $M_\alpha$  undergoes a continuous or a discontinuous phase transition from an inactive absorbing state with  $M_\alpha=0$  to an active state with  $M_\alpha > 0$ . All hypersurfaces of phase transitions meet in the multicritical point where all temperatures are zero. All homogeneous states are globally stable because the evolution of the total particle density of homogeneous states in time is given by  $d(\sum_\alpha M_\alpha)/dt = -\sum_\alpha \lambda_\alpha \tau_\alpha M_\alpha - \frac{1}{2} \sum_{\alpha, \beta} \lambda_\alpha g_{\alpha\beta} M_\alpha M_\beta$ . Thus, all solutions of the mean field equations of motion are bounded to a finite region in the space of positive  $M_\alpha$  as long as the stability condition mentioned in the foregoing paragraph holds.

In the following we focus on the effect of fluctuations on the scaling behavior of correlation and response functions in the vicinity of the multicritical point where the strongly relevant parameters  $\{\tau_\alpha\}$  are small. In order to apply field-theoretic methods and the renormalization group equation in conjunction with an  $\varepsilon$ -expansion about the upper critical dimension,<sup>(32-36)</sup> it is convenient to use the path-integral representation of the underlying stochastic processes  $n(\mathbf{x}, t) = \{n_\alpha(\mathbf{x}, t)\}$ .<sup>(36-39)</sup> With the imaginary-valued response fields denoted by  $\tilde{n}(\mathbf{x}, t) = \{\tilde{n}_\alpha(\mathbf{x}, t)\}$ , the generating functional of the connected response and correlation functions, the Green functions, takes the form

$$\mathcal{W}[h, \tilde{h}] = \ln \int \mathcal{D}[\tilde{n}, n] \exp \left( -\mathcal{J}[\tilde{n}, n] + \int d^d x \int dt (hn + \tilde{h}\tilde{n}) \right) \quad (4)$$

The response fields  $\tilde{n}(\mathbf{x}, t)$  correspond to the conjugated auxiliary variables of the operator formulation of statistical dynamics by Kawasaki<sup>(40)</sup> and Martin, Siggia, Rose.<sup>(41)</sup> The dynamic functional  $\mathcal{J}[\tilde{n}, n]$  and the functional measure  $\mathcal{D}[\tilde{n}, n]$ , which in symbolic notation is proportional to  $\prod_{\mathbf{x}, t} (d\tilde{n}(\mathbf{x}, t) dn(\mathbf{x}, t))$ , is understood to be defined using a prepoint (Ito) discretization with respect to time. The prepoint discretization leads to the causality rule  $\theta(t \leq 0) = 0$  in response functions which forbids response propagator loops in the diagrammatical perturbation expansion.<sup>(36, 39)</sup>

The Langevin equations (1)–(3) are recast as a dynamic functional

$$\begin{aligned} \mathcal{J} &= \int dt d^d x \left( \sum_{\alpha} \tilde{n}_{\alpha} (\partial_t - \lambda_{\alpha} \nabla^2 + R_{\alpha}(n)) n_{\alpha} - \sum_{\alpha\beta} \tilde{n}_{\alpha} D_{\alpha\beta}(n) \tilde{n}_{\beta} \right) \\ &= \int dt d^d x \sum_{\alpha} \lambda_{\alpha} \tilde{n}_{\alpha} \left( \lambda_{\alpha}^{-1} \partial_t + \tau_{\alpha} - \nabla^2 + \frac{1}{2} \left( \sum_{\beta} g_{\alpha\beta} n_{\beta} - g_{\alpha} \tilde{n}_{\alpha} \right) \right) n_{\alpha} \quad (5) \end{aligned}$$

In the second line we have neglected subleading terms. Correlation and response functions can now be expressed as functional averages of monomials of the  $n_{\alpha}$  and  $\tilde{n}_{\alpha}$  with weight  $\exp(-\mathcal{J})$ . A glance at Eq. (4) shows that the responses are defined with respect to additional local particle sources  $\tilde{h}_{\alpha}(\mathbf{x}, t) \geq 0$  in the Langevin equations (1). A rescaling of the fields  $n_{\alpha} \rightarrow l_{\alpha} n_{\alpha}$ ,  $\tilde{n}_{\alpha} \rightarrow l_{\alpha}^{-1} \tilde{n}_{\alpha}$  leaves the functional  $\mathcal{J}$  forminvariant but transforms the coupling constants  $g_{\alpha\beta} \rightarrow l_{\beta} g_{\alpha\beta}$  and  $g_{\alpha} \rightarrow l_{\alpha}^{-1} g_{\alpha}$ . Thus, invariant coupling constants are given by  $f_{\alpha\beta} = g_{\alpha\beta} g_{\beta}$ . A suitable rescaling is defined by the conditions  $g_{\alpha\alpha} = g_{\alpha}$ . We choose this normalization and denote the rescaled fields by  $s_{\alpha} \sim n_{\alpha}$  and  $\tilde{s}_{\alpha} \sim \tilde{n}_{\alpha}$ .

The scaling by a suitable mesoscopic length and time scale,  $\mu^{-1}$  and  $(\lambda\mu^2)^{-1}$  (with  $\lambda_{\alpha} \sim \lambda$ ) respectively, leads to  $\tilde{s}_{\alpha} \sim s_{\alpha} \sim \mu^{d/2}$ ,  $f_{\alpha\beta} \sim \mu^{\varepsilon}$  where  $\varepsilon = 4 - d$ . Hence  $d_c = 4$  is the upper critical dimension. Now it is easy to show that all neglected possible subleading terms in the expansion of  $R_{\alpha}(n)$  and the noise correlation, as well as higher gradient terms in the Langevin equation (1) and non-Gaussian and non-Markovian noise correlations, have coupling constants with negative  $\mu$ -dimensions near the upper critical dimension. Therefore, under the renormalization group flow, they are renormalized to zero and we can safely neglect them because we are interested in the leading universal critical behavior. These couplings can be reintroduced if one is interested in corrections to scaling. In a renormalized field theory context,<sup>(32, 33)</sup> the  $\mu$ -dimensions of the coupling constants are equal to the so called naive or engineering dimensions. All coupling constants with positive naive dimensions (the relevant couplings with respect to the Gaussian fixed point as the starting point of the perturbation expansion) need a renormalization because the corresponding vertex functions

develop primitive divergencies in perturbation theory. However, no further primitive divergencies are generated in other vertex functions. Thus the field theory based on the functional  $\mathcal{J}$  (Eq. (5)) is renormalizable and the calculated scaling properties are universal for the full class of MDP-processes.

### III. RENORMALIZATION AND ONE-LOOP CALCULATION

Now we are in a position to develop the perturbation theory in the inactive phase with all the  $\tau_\alpha > 0$ . To begin with we separate the anharmonic “interaction” terms from the dynamic functional  $\mathcal{J}$ , Eq. (5), and retain only the harmonic ones:

$$\mathcal{J}_0 = \int dt d^d x \sum_\alpha (\tilde{s}_\alpha (\partial_t + \lambda_\alpha (\tau_\alpha - \nabla^2)) s_\alpha - \tilde{h}_\alpha \tilde{s}_\alpha - h_\alpha s_\alpha) \quad (6)$$

Here we have included the external sources  $\tilde{h}_\alpha$  and  $h_\alpha$ . The Gaussian path integral  $\int \mathcal{D}[\tilde{s}, s] \exp(-\mathcal{J}_0) = \exp((h, G\tilde{h}))$  involves *only* non-negative fields  $s_\alpha(\mathbf{x}, t) \geq 0$  as long as  $\tilde{h}_\alpha(\mathbf{x}, t) \geq 0$ . This may be seen as follows. Integrating out the response fields  $\tilde{s}_\alpha$  leads to a product of  $\delta$ -functions  $\delta(\partial_t s_\alpha + \lambda_\alpha (\tau_\alpha - \nabla^2) s_\alpha - \tilde{h}_\alpha)$ . The solutions of the corresponding differential equations subject to the initial conditions  $s_\alpha(\mathbf{x}, t = -\infty) = 0$  are given by  $s_\alpha(\mathbf{x}, t) = \int d^d x' dt' G_\alpha(\mathbf{x} - \mathbf{x}', t - t') \tilde{h}_\alpha(\mathbf{x}', t')$  with positive kernels  $G_\alpha(\mathbf{x}, t) = \theta(t) \times (4\pi\lambda_\alpha t)^{-d/2} \exp(-\lambda_\alpha \tau_\alpha - \mathbf{x}^2/(4\lambda_\alpha t))$ . They yield the propagators for the Fourier transformed fields  $s_\alpha(\mathbf{q}, t) = \int d^d x s_\alpha(\mathbf{x}, t) \exp(-i\mathbf{q} \cdot \mathbf{x})$  etc., as

$$\begin{aligned} \langle s_\alpha(\mathbf{q}, t) \tilde{s}_\beta(\mathbf{q}', t') \rangle_0 &= (2\pi)^d \delta(\mathbf{q} + \mathbf{q}') \delta_{\alpha, \beta} G_\alpha(\mathbf{q}, t - t') \\ G_\alpha(\mathbf{q}, t) &= \theta(t) \exp(-\lambda_\alpha (\tau_\alpha + q^2) t) \end{aligned} \quad (7)$$

where the Heaviside theta-function is defined with  $\theta(t=0) = 0$  following from the Ito-discretization of the path-integral and ensuring causality.

Besides the propagators, the anharmonic coupling terms in  $\mathcal{J}$ , Eq. (5), define the elements of the graphical perturbation expansion. They are depicted in Fig. 1 where an arrow marks a  $\tilde{s}$ -leg and we draw diagrams with the arrows always directed to the left (ascending time ordering from right to left).

Figure 1 shows that the color of a  $\tilde{s}$ -leg on the left side of a vertex is not annihilated: at least one  $s$ -leg on the right side displays the same color. Thus, going from left to right, i.e., backward in time, through a diagram, colors have only sources and no sinks. This property is a consequence of the existence of absorbing states in the model. The perturbation expansion



$$\begin{aligned}
 & \begin{array}{c} \mathbf{q} \\ \leftarrow \leftarrow \leftarrow \\ t \quad \quad \quad t' \\ \alpha \end{array} = G_\alpha(\mathbf{q}, t - t') \\
 & \begin{array}{c} \alpha \\ \swarrow \quad \rightarrow \\ \alpha \quad \quad \alpha \end{array} = \lambda_\alpha g_\alpha \\
 & \begin{array}{c} \quad \quad \gamma \\ \swarrow \quad \searrow \\ \alpha \quad \quad \beta \end{array} = -\frac{\lambda_\alpha}{2}(\delta_{\alpha,\gamma} g_{\alpha\beta} + \delta_{\alpha,\beta} g_{\alpha\gamma})
 \end{aligned}$$

Fig. 1. Elements of the graphical perturbation expansion.

of a translationally invariant field theory can be analyzed by the calculation of the vertex functions  $\Gamma_{\{\alpha\}, \{\beta\}}(\{\mathbf{q}, \omega\})$  corresponding to the one-particle irreducible amputated diagrams. Here the sets  $\{\alpha\}$  and  $\{\beta\}$  denote the colors of the amputated outer  $\tilde{s}$ -legs and  $s$ -legs respectively. Going backward in time we conclude from the conservation property of the  $\tilde{s}$ -colors that the colors of the set  $\{\alpha\}$  must appear as a subset of  $\{\beta\}$ . Therefore the only nonzero two and three point vertex functions are  $\Gamma_{\alpha, \alpha}$ ,  $\Gamma_{\alpha\alpha, \alpha}$ , and  $\Gamma_{\alpha, \alpha\beta}$ . Moreover, another property follows directly from color conservation: the vertex function  $\Gamma_{\{\alpha\}, \{\beta\}}$  does not depend on parameters  $\lambda$ ,  $\tau$ , and  $g$  with colors other than the ones of the vertex functions itself. Thus,  $\Gamma_{\alpha, \alpha}$ ,  $\Gamma_{\alpha\alpha, \alpha}$ , and  $\Gamma_{\alpha, \alpha\alpha}$  are only functions of the parameters  $\lambda_\alpha$ ,  $\tau_\alpha$ , and  $g_{\alpha\alpha} = g_\alpha$  and are in particular independent of the interspecies couplings  $g_{\alpha\beta}$  with  $\alpha \neq \beta$ . Therefore these vertex functions are the same as the corresponding functions of the well analyzed one-species Gribov process. To calculate them one can set the interspecies couplings  $g_{\alpha\beta} = 0$ . In this case the model shows time reflection invariance called rapidity reversal in Reggeon field theory  $\tilde{s}_\alpha(t) \leftrightarrow -s_\alpha(-t)$  from which follows the equality  $\Gamma_{\alpha\alpha, \alpha} = -\Gamma_{\alpha, \alpha\alpha}$ .

$$\begin{aligned}
 & \begin{array}{c} \alpha \\ \leftarrow \text{---} \text{---} \text{---} \text{---} \alpha \end{array} = -\Gamma_{\alpha, \alpha} \\
 & \begin{array}{c} \alpha \\ \leftarrow \text{---} \text{---} \text{---} \alpha \\ \alpha \end{array} = - \begin{array}{c} \alpha \\ \swarrow \quad \searrow \\ \alpha \quad \quad \alpha \end{array} = -\Gamma_{\alpha, \alpha\alpha} \\
 & \begin{array}{c} \alpha \\ \leftarrow \text{---} \text{---} \text{---} \beta \\ \quad \quad \gamma \end{array} = -\frac{1}{2}(\delta_{\alpha,\gamma} \Gamma_{\alpha, \alpha\beta} + \delta_{\alpha,\beta} \Gamma_{\alpha, \alpha\gamma})
 \end{aligned}$$

Fig. 2. Primitively divergent vertex functions.

Now we are ready to consider the renormalization of the model. It is known that the perturbation expansion of a field theory with a momentum cutoff  $\Lambda$  develops divergencies if  $\Lambda \rightarrow \infty$ .<sup>(32,33)</sup> If the model is renormalizable one can absorb all these “primitive divergencies” order by order in a loop expansion into a suitable reparametrization of the parameters. Absorbing the primitive divergencies regularizes the full model. The primitively divergent vertex functions have nonnegative naive  $\mu$ -dimensions. Here, they are  $\Gamma_{\alpha,\alpha} \sim \mu^2$  and  $\Gamma_{\alpha\alpha,\alpha}, \Gamma_{\alpha,\alpha\beta} \sim \mu^\varepsilon$ , logarithmic at the upper critical dimension, (Fig. 2).

Taking into account the general properties of the vertex functions found in the foregoing paragraph, we see that the following renormalization scheme renders the theory finite

$$\begin{aligned}
 s_\alpha &\rightarrow \hat{s}_\alpha = \sqrt{Z_s^{(\alpha)}} s_\alpha, & \tilde{s}_\alpha &\rightarrow \hat{\tilde{s}}_\alpha = \sqrt{Z_s^{(\alpha)}} \tilde{s}_\alpha, & \lambda_\alpha &\rightarrow \hat{\lambda}_\alpha = \frac{Z_\lambda^{(\alpha)}}{Z_s^{(\alpha)}} \lambda_\alpha \\
 \tau_\alpha &\rightarrow \hat{\tau}_\alpha = \frac{Z_\tau^{(\alpha)}}{Z_\lambda^{(\alpha)}} \tau_\alpha, & f_{\alpha\beta} &\rightarrow \hat{f}_{\alpha\beta} = G_\varepsilon^{-1} \mu^\varepsilon \frac{Z_u^{(\alpha\beta)}}{Z_s^{(\beta)} Z_\lambda^{(\alpha)} Z_\lambda^{(\beta)}} u_{\alpha\beta}
 \end{aligned} \tag{8}$$

Here  $G_\varepsilon = \Gamma(1 + \varepsilon/2)/(4\pi)^{d/2}$  is a convenient constant. Instead of a momentum-cutoff regularization, we use dimensional regularization and minimal renormalization in the following. From the discussion above we learn that the renormalization factors  $Z_i^{(\alpha)}$  with  $i = (s, \tau, \lambda)$  and  $Z_u^{(\alpha)} := Z_u^{(\alpha\alpha)}$ , which are determined from  $\Gamma_{\alpha,\alpha}$  and  $\Gamma_{\alpha\alpha,\alpha}$ , are already known from the one-species Gribov process and depend only on  $u_\alpha := u_{\alpha\alpha}$ . Thus  $Z_i^{(\alpha)} = Z_i(u_\alpha, \varepsilon)$ . The new renormalization factors  $Z_u^{(\alpha\beta)} = Z_{\text{int}}(\{u\}, \{\lambda\}, \varepsilon)$  with  $\alpha \neq \beta$  stem from the interspecies couplings. They depend only on the couplings  $u_\alpha, u_\beta, u_{\alpha\beta}, u_{\beta\alpha}$ , and the ratio  $\lambda_\alpha/\lambda_\beta$ .

We will now explicitly calculate the renormalizations to one-loop order. The primitively divergent one-loop diagrams are shown in Fig. 3.

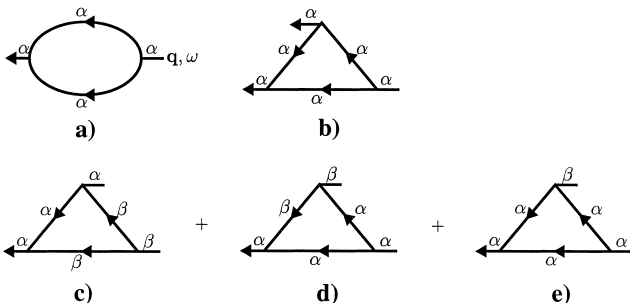


Fig. 3. Primitively divergent one-loop diagrams.

Using dimensional regularization, we express the contribution of the self-energy diagram, Fig. 3(a), as a function of external momentum and frequency,  $\mathbf{q}$  and  $\omega$ :

$$\begin{aligned} 3(a) &= -\frac{(\lambda_\alpha g_\alpha)^2}{2} \int_{\mathbf{p}} \frac{1}{i\omega + 2\lambda_\alpha \tau_\alpha + \lambda_\alpha \mathbf{p}^2 + \lambda_\alpha (\mathbf{p} + \mathbf{q})^2} \\ &= \frac{G_\varepsilon}{4\varepsilon} \tau_\alpha^{-\varepsilon/2} \lambda_\alpha g_\alpha^2 \left( \frac{4\tau_\alpha}{2-\varepsilon} + \frac{i\omega}{\lambda_\alpha} + \frac{\mathbf{q}^2}{2} \right) + \dots \end{aligned} \quad (9)$$

Here, we have retained only terms linear in  $\omega$  and  $\mathbf{q}^2$ . These are the terms that display poles in  $\varepsilon = 4 - d > 0$ .

To determine the primitive divergencies of the vertex functions we can set external momenta and frequencies to zero and use equal temperatures  $\tau_\alpha = \tau > 0$  as infrared regularisers. The contributions of the three diagrams Fig. 3(c,d,e) add to

$$\begin{aligned} 3(c) + 3(d) + 3(e) &= \frac{\lambda_\alpha g_{\alpha\beta}}{2} \left( \frac{\lambda_\alpha g_{\alpha\beta}}{2} \lambda_\beta g_\beta \int_{\mathbf{p}} \frac{1}{2\lambda_\beta (\lambda_\alpha + \lambda_\beta) (\tau + \mathbf{p}^2)^2} \right. \\ &\quad + \frac{\lambda_\beta g_{\beta\alpha}}{2} \lambda_\alpha g_\alpha \int_{\mathbf{p}} \frac{1}{2\lambda_\alpha (\lambda_\alpha + \lambda_\beta) (\tau + \mathbf{p}^2)^2} \\ &\quad \left. + \lambda_\alpha^2 g_{\alpha\alpha} g_\alpha \int_{\mathbf{p}} \frac{1}{(2\lambda_\alpha)^2 (\tau + \mathbf{p}^2)^2} \right) \\ &= \frac{G_\varepsilon}{4\varepsilon} \tau^{-\varepsilon/2} \lambda_\alpha g_{\alpha\beta} \left( \frac{\lambda_\alpha g_{\alpha\beta} g_\beta + \lambda_\beta g_{\beta\alpha} g_\alpha + g_{\alpha\alpha} g_\alpha}{\lambda_\alpha + \lambda_\beta} \right) \end{aligned} \quad (10)$$

From the zero-loop contributions and the results of our short calculation, Eqs. (9), (10), we obtain the (as yet unrenormalized) one-loop vertex functions to the desired order in  $\omega$  and  $\mathbf{q}^2$ :

$$\begin{aligned} \Gamma_{\alpha\alpha} &= i\omega \left( 1 - \frac{G_\varepsilon}{4\varepsilon} g_{\alpha\alpha} g_\alpha \tau_\alpha^{-\varepsilon/2} \right) + \lambda_\alpha \mathbf{q}^2 \left( 1 - \frac{G_\varepsilon}{8\varepsilon} g_{\alpha\alpha} g_\alpha \tau_\alpha^{-\varepsilon/2} \right) \\ &\quad + \lambda_\alpha \tau_\alpha \left( 1 - \frac{G_\varepsilon}{2\varepsilon(1-\varepsilon/2)} g_{\alpha\alpha} g_\alpha \tau_\alpha^{-\varepsilon/2} \right) + \dots \end{aligned} \quad (11)$$

and

$$\Gamma_{\alpha,\alpha\beta} = \lambda_\alpha g_{\alpha\beta} \left( 1 - \frac{G_\varepsilon}{2\varepsilon} \left( \frac{\lambda_\alpha g_{\alpha\beta} g_\beta + \lambda_\beta g_{\beta\alpha} g_\alpha}{\lambda_\alpha + \lambda_\beta} + g_{\alpha\alpha} g_\alpha \right) \tau^{-\varepsilon/2} \right) \quad (12)$$

An explicit calculation of diagram (b) of Fig. 3 demonstrates that indeed  $\Gamma_{\alpha\alpha,\alpha} = -\Gamma_{\alpha,\alpha\alpha}$  if  $g_{\alpha\alpha} = g_\alpha$ .

To absorb the  $\varepsilon$ -poles in the renormalization  $Z$ -factors we note that the vertex functions are renormalized by the scheme Eq. (8) as

$$\Gamma_{\alpha_1 \dots \alpha_n} \rightarrow \hat{\Gamma}_{\alpha_1 \dots \alpha_n} = (Z_s^{(\alpha_1)} \dots Z_s^{(\alpha_n)})^{-1/2} \Gamma_{\alpha_1 \dots \alpha_n} \quad (13)$$

Using again the renormalization scheme Eq. (8), we find the renormalized vertex functions from Eqs. (11), (12) as

$$\begin{aligned} \Gamma_{\alpha\alpha} = i\omega & \left( Z_s^{(\alpha)} - \frac{u_\alpha}{4\varepsilon} (\mu/\tau_\alpha)^{\varepsilon/2} \right) + \lambda_\alpha \mathbf{q}^2 \left( Z_\lambda^{(\alpha)} - \frac{u_\alpha}{8\varepsilon} (\mu/\tau_\alpha)^{\varepsilon/2} \right) \\ & + \lambda_\alpha \tau_\alpha \left( Z_\tau^{(\alpha)} - \frac{u_\alpha}{2\varepsilon(1-\varepsilon/2)} (\mu/\tau_\alpha)^{\varepsilon/2} \right) + \dots \end{aligned} \quad (14)$$

and

$$\Gamma_{\alpha,\alpha\beta} \Gamma_{\beta,\beta\beta} = G_\varepsilon^{-1} \mu^\varepsilon \lambda_\alpha^2 u_{\alpha\beta} \left( Z_u^{(\alpha\beta)} - \frac{1}{2\varepsilon} \left( \frac{\lambda_\alpha u_{\alpha\beta} + \lambda_\beta u_{\beta\alpha}}{\lambda_\alpha + \lambda_\beta} + u_\alpha + 2u_\beta \right) (\mu/\tau)^{\varepsilon/2} \right) \quad (15)$$

Therefore the vertex functions become finite by choosing

$$\begin{aligned} Z_s^{(\alpha)} &= 1 + \frac{u_\alpha}{4\varepsilon}, & Z_\lambda^{(\alpha)} &= 1 + \frac{u_\alpha}{8\varepsilon}, & Z_\tau^{(\alpha)} &= 1 + \frac{u_\alpha}{2\varepsilon} \\ Z_u^{(\alpha\beta)} &= 1 + \frac{1}{2\varepsilon} \left( \frac{\lambda_\alpha u_{\alpha\beta} + \lambda_\beta u_{\beta\alpha}}{\lambda_\alpha + \lambda_\beta} + u_\alpha + 2u_\beta \right) \end{aligned} \quad (16)$$

up to higher orders in the coupling constants  $u$ . As anticipated, for  $\alpha = \beta$  we have found the well known renormalization factors of the Reggeon field theory.

#### IV. RENORMALIZATION GROUP ANALYSIS AND ASYMPTOTIC SCALING

Next we will explore the scaling properties of multicolored directed percolation. Scaling properties describe how physical quantities will transform under a change of length scales. At the end of Chapter II we have introduced the arbitrary mesoscopic length scale  $\mu$ . The freedom in the choice of  $\mu$ , keeping the unrenormalized bare parameters  $\{\hat{\tau}_\alpha, \hat{\lambda}_\alpha, \hat{g}_\alpha, \hat{g}_{\alpha\beta}\}$ , and—in cutoff regularization—the momentum cutoff  $\Lambda$  fixed, can be used

to derive in a routine fashion the renormalization group (RG) equation for the connected correlation and response functions, the Green functions

$$G^{\{N, \tilde{N}\}}(\{\mathbf{x}, t\}) = \left\langle \prod_{\alpha} \left( \prod_{i=1}^{N_{\alpha}} s(\mathbf{x}_i^{(\alpha)}, t_i^{(\alpha)}) \prod_{j=N_{\alpha}+1}^{N_{\alpha}+\tilde{N}_{\alpha}} \tilde{s}(\mathbf{x}_j^{(\alpha)}, t_j^{(\alpha)}) \right) \right\rangle^{conn} \quad (17)$$

We denote  $\mu$ -derivatives at fixed bare parameters by  $\partial_{\mu}|_0$ . From  $\mu \partial_{\mu}|_0 \overset{\circ}{G}^{\{N, \tilde{N}\}} = 0$  and the renormalization scheme Eq. (8), which leads to  $\overset{\circ}{G}^{\{N, \tilde{N}\}} = \prod_{\alpha} (Z_s^{(\alpha)})^{(N_{\alpha}+\tilde{N}_{\alpha})/2} G^{\{N, \tilde{N}\}}$ , we then find the RG equations

$$\left[ \mathcal{D}_{\mu} + \sum_{\alpha} \frac{N_{\alpha} + \tilde{N}_{\alpha}}{2} \gamma_s^{(\alpha)} \right] G^{\{N, \tilde{N}\}} = 0 \quad (18)$$

with the renormalization group differential operator

$$\mathcal{D}_{\mu} = \mu \partial_{\mu} + \sum_{\alpha} (\zeta_{\alpha} \lambda_{\alpha} \partial_{\lambda_{\alpha}} + \kappa_{\alpha} \tau_{\alpha} \partial_{\tau_{\alpha}} + \beta_{\alpha} \partial_{u_{\alpha}}) + \sum_{\alpha \neq \beta} \beta_{\alpha\beta} \partial_{u_{\alpha\beta}} \quad (19)$$

Here we have introduced the Gell–Mann–Low functions

$$\begin{aligned} \zeta_{\alpha} &= \mu \partial_{\mu}|_0 \ln \lambda_{\alpha} = \gamma_s^{(\alpha)} - \gamma_{\lambda}^{(\alpha)} \\ \kappa_{\alpha} &= \mu \partial_{\mu}|_0 \ln \tau_{\alpha} = \gamma_{\lambda}^{(\alpha)} - \gamma_{\tau}^{(\alpha)} \\ \beta_{\alpha\beta} &= \mu \partial_{\mu}|_0 u_{\alpha\beta} = (-\varepsilon + \gamma_s^{(\beta)} + \gamma_{\lambda}^{(\alpha)} + \gamma_{\lambda}^{(\beta)} - \gamma_u^{(\alpha\beta)}) u_{\alpha\beta} \end{aligned} \quad (20)$$

with  $\beta_{\alpha} = \beta_{\alpha\alpha}$ , and the Wilson functions

$$\gamma_i^{(r)} = \mu \partial_{\mu}|_0 \ln Z_i^{(r)}, \quad i = s, \lambda, \tau, u, \quad r = \alpha, \alpha\beta \quad (21)$$

The RG equations (18) can be solved in terms of a single flow parameter  $l$  using characteristics. Following this method, flowing parameters are defined by the characteristic equations

$$\begin{aligned} l \frac{d}{dl} \bar{u}_{\alpha\beta}(l) &= \beta_{\alpha\beta}(\bar{u}(l)), & \bar{u}_{\alpha\beta}(1) &= u_{\alpha\beta} \\ l \frac{d}{dl} \bar{\lambda}_{\alpha}(l) &= \zeta_{\alpha}(\bar{u}(l)) \bar{\lambda}_{\alpha}(l) = \zeta(\bar{u}_{\alpha}(l)) \bar{\lambda}_{\alpha}(l), & \bar{\lambda}_{\alpha}(1) &= \lambda_{\alpha} \\ l \frac{d}{dl} \bar{\tau}_{\alpha}(l) &= \kappa_{\alpha}(\bar{u}(l)) \bar{\tau}_{\alpha}(l) = \kappa(\bar{u}_{\alpha}(l)) \bar{\tau}_{\alpha}(l), & \bar{\tau}_{\alpha}(1) &= \tau_{\alpha} \end{aligned} \quad (22)$$

and the RG equations (18) of the Green functions become

$$\left[ l \frac{d}{dl} + \sum_{\alpha} \frac{N_{\alpha} + \tilde{N}_{\alpha}}{2} \gamma(\bar{u}_{\alpha}(l)) \right] G^{\{N, \tilde{N}\}}(\{\mathbf{x}, t\}, \{\bar{\tau}(l)\}, \{\bar{u}(l)\}, \{\bar{\lambda}(l)\}, l\mu) = 0 \quad (23)$$

Here, the functions  $\gamma = \gamma_s^{(\alpha)}$ ,  $\zeta = \zeta_{\alpha}$ , and  $\kappa = \kappa_{\alpha}$  are independent of the color  $\alpha$  and the interspecies couplings. The flow equations (22) describe how the parameters transform if we change the momentum scale  $\mu$  according to  $\mu \rightarrow \bar{\mu}(l) = l\mu$ . Being interested in the infrared (IR) behavior of the theory, we must study the limit  $l \rightarrow 0$ . In general we expect that in this IR limit the coupling constants  $\bar{u}_{\alpha\beta}(l)$  flow to a stable fixed point  $u_{\alpha\beta,*}$  according to the first set of Eq. (22). In particular, the intraspecies couplings  $\bar{u}_{\alpha\alpha} = \bar{u}_{\alpha}$  then flow to a color independent fixed point  $\bar{u}_{\alpha}(0) = u_{*}$  because  $\beta_{\alpha}(\bar{u}) = \beta(\bar{u}_{\alpha})$ , and this Gell-Mann–Low function  $\beta$  is equal to the corresponding function known from the one-species Gribov process. Thus, the fixed point value  $u_{*}$  is independent of any coupling to other species.

The solutions of the second and third set of the flow equations (23) are readily found in terms of the functions  $\bar{u}_{\alpha}(l)$ . In the IR limit  $l \ll 1$  they have the scaling form

$$\bar{\lambda}_{\alpha}(l) = l^{z-2}(1 + O(l)) A_{\lambda}^{(\alpha)} \lambda_{\alpha}, \quad \bar{\tau}_{\alpha}(l) = l^{2-1/\nu}(1 + O(l)) A_{\tau}^{(\alpha)} \tau_{\alpha} \quad (24)$$

where the  $A_i^{(\alpha)}$  are nonuniversal flavor dependent amplitude factors. The scaling exponents

$$z = 2 + \zeta(u_{*}), \quad \nu = \frac{1}{2 - \kappa(u_{*})} \quad (25)$$

are already known from directed percolation. The ratios of the kinetic coefficients for different flavors  $\bar{\lambda}_{\alpha}(l)/\bar{\lambda}_{\beta}(l)$  are nonuniversal even in the IR limit.

From Eq. (23) we find the solution in the IR limit as

$$G^{\{N, \tilde{N}\}}(\{\mathbf{x}, t\}, \{\bar{\tau}\}, \{\mathbf{u}\}, \{\lambda\}, \mu) = l^{(N+\tilde{N})\eta/2}(1 + O(l)) \prod_{\alpha} (A_s^{(\alpha)})^{(N_{\alpha} + \tilde{N}_{\alpha})/2} \times G^{\{N, \tilde{N}\}}(\{\mathbf{x}, t\}, \{l^{2-1/\nu} A_{\tau} \tau\}, \{u_{*}\}, \{l^{z-2} A_{\lambda} \lambda\}, l\mu) \quad (26)$$

with  $N = \sum_{\alpha} N_{\alpha}$ ,  $\tilde{N} = \sum_{\alpha} \tilde{N}_{\alpha}$ , and the anomalous field scaling exponent

$$\eta = \gamma(u_{*}) \quad (27)$$

of DP. Henceforth we omit the amplitude factors  $A_i^{(\alpha)}$ . In order to obtain the scaling behavior of the Green functions we supplement the renormalization group result with a dimensional analysis. Dimensional analysis in space and time gives

$$G^{\{N, \tilde{N}\}}(\{\mathbf{x}, t\}, \{\tau\}, \{\mathbf{u}\}, \{\lambda\}, \mu) = \mu^{(N+\tilde{N})d/2} G^{\{N, \tilde{N}\}}(\{\mu\mathbf{x}, \lambda\mu^2 t\}, \{\tau/\mu^2\}, \{\mathbf{u}\}, \{c\}, 1) \tag{28}$$

where  $c_\alpha = \lambda_\alpha/\lambda$ . By combining Eqs. (26) and (28) we now get the asymptotic scaling form of the Green functions

$$G^{\{N, \tilde{N}\}}(\{\mathbf{x}, t\}, \{\tau\}) = l^{(N+\tilde{N})(d+\eta)/2} G^{\{N, \tilde{N}\}}(\{l\mathbf{x}, l^z t\}, \{\tau/l^{1/\nu}\}) \tag{29}$$

An important consequence of the last equation is that all scaling properties of the DP processes remain unaffected by the introduction of different colors. Moreover, all intraspecies Green functions are completely independent of the coupling between the species. These are consequences from the absorbing state conditions for each species.

However, the interspecies coupling constants  $u_{\alpha\beta}$  with  $\alpha \neq \beta$  determine the properties of the interspecies scaling functions. Therefore we will now consider the consequences of the flow equations for these parameters. For this purpose we need the Gell-Mann–Low functions  $\beta_{\alpha\beta}$  explicitly. From the last of Eq. (20) we know that each of these functions begins with the zero-loop term  $-\epsilon u_{\alpha\beta}$  and that the higher order terms are determined by the Wilson functions. These functions, the logarithmic derivatives of the  $Z$ -factors, are given by  $\gamma = \mu \partial_\mu|_0 \ln Z = \sum_{\alpha\beta} \beta_{\alpha\beta} \partial_{u_{\alpha\beta}} \ln Z$ . In minimal renormalization the  $Z$ -factors have a pure Laurent expansion with respect to  $\epsilon$ :  $Z = 1 + Y^{(1)}/\epsilon + Y^{(2)}/\epsilon^2 + \dots$ . Thus recursively in the loop expansion the Wilson functions also have a pure Laurent expansion and, because they are finite for  $\epsilon \rightarrow 0$ , this expansion reduces to the constant term, i.e., all  $\epsilon$ -poles have to cancel in the logarithmic derivation. Hence, we obtain the Wilson functions simply from the formula  $\gamma = -\sum_{\alpha\beta} u_{\alpha\beta} \partial_{u_{\alpha\beta}} Y^{(1)}$ . Now it is easy to get these functions from the one loop results of the  $Z$ -factors Eq. (16). We find to this order

$$\begin{aligned} \gamma_s^{(\alpha)} &= -\frac{u_\alpha}{4}, & \gamma_\lambda^{(\alpha)} &= -\frac{u_\alpha}{8}, & \gamma_\tau^{(\alpha)} &= -\frac{u_\alpha}{2} \\ \gamma_u^{(\alpha\beta)} &= -\frac{1}{2} \left( \frac{\lambda_\alpha u_{\alpha\beta} + \lambda_\beta u_{\beta\alpha}}{\lambda_\alpha + \lambda_\beta} + u_\alpha + 2u_\beta \right) \end{aligned} \tag{30}$$

from which one finds the Gell-Mann–Low functions Eq. (20) as

$$\beta_{\alpha\beta} = \left( -\varepsilon + \frac{3u_\alpha}{8} + \frac{5u_\beta}{8} + \frac{\lambda_\alpha u_{\alpha\beta} + \lambda_\beta u_{\beta\alpha}}{2(\lambda_\alpha + \lambda_\beta)} \right) u_{\alpha\beta} \quad (31)$$

The Gell-Mann–Low functions of the intraspecies couplings, the “diagonal”  $\beta$ -functions, follow as  $\beta_\alpha = (-\varepsilon + 3u_\alpha/2) u_\alpha$  giving the stable fixed point values  $u_{\alpha*} = u_* = 2\varepsilon/3$ . With the help of Eqs. (20), (25), (27), this stable fixed point leads to the well known one loop order DP exponents  $z = 2 - \varepsilon/12$ ,  $\nu = 1/2 + \varepsilon/16$ , and  $\eta = -\varepsilon/6$ . Using the fixed point values  $u_{\alpha*}$  to obtain the Gell-Mann–Low functions of the interspecies couplings of a pair of colors  $\alpha \neq \beta$ , we get

$$\beta_{\alpha\beta} = \left( -\frac{\varepsilon}{3} + \frac{\lambda_\alpha u_{\alpha\beta} + \lambda_\beta u_{\beta\alpha}}{2(\lambda_\alpha + \lambda_\beta)} \right) u_{\alpha\beta} \quad (32)$$

In addition to the unstable decoupled fixed point values  $u_{\alpha\beta*} = u_{\beta\alpha*} = 0$ , the equations  $\beta_{\alpha\beta} = \beta_{\beta\alpha} = 0$  are solved by a line of fixed points

$$\frac{2\lambda_\alpha}{\lambda_\alpha + \lambda_\beta} u_{\alpha\beta*} + \frac{2\lambda_\beta}{\lambda_\alpha + \lambda_\beta} u_{\beta\alpha*} = \frac{4\varepsilon}{3} \quad (33)$$

This equation is the key result of this Section. Clearly, however, the one-loop calculation cannot give us any information whether the degeneracy of all the points on this line is fundamental to our model or is remedied by higher loop corrections. Thus, we must proceed to two-loop order, as done in the following chapter. We note, however, that there are two special points of unidirectional coupling on the fixed line:

$$\begin{aligned} u_{\alpha\beta*} = 0, \quad u_{\beta\alpha*} &= \frac{2(\lambda_\alpha + \lambda_\beta)}{3\lambda_\beta} \varepsilon & \text{and} \\ u_{\beta\alpha*} = 0, \quad u_{\alpha\beta*} &= \frac{2(\lambda_\alpha + \lambda_\beta)}{3\lambda_\alpha} \varepsilon \end{aligned} \quad (34)$$

In the next chapter we will show that for colors with the same flavor, meaning  $\lambda_\alpha = \lambda_\beta$ , these two points are indeed the only stable points on the line. Because  $u_{\alpha\beta*} = 0$  or  $u_{\beta\alpha*} = 0$  are fixed point values at any loop order, we conjecture that unidirectionality is generic for the asymptotic behavior of coupled DP processes, irrespective of whether colors belong to different flavors or not.



## V. TWO-LOOP RESULTS AND CROSSOVER TO UNIDIRECTIONALITY

From the analysis in the foregoing chapters we know that the interaction of two species does not depend on the existence of other ones. Thus, we consider in this chapter a coupled model of two species of colors  $\alpha = 1$  and 2 with the same flavor, i.e., equal kinetic coefficients  $\lambda_\alpha = \lambda$  and intra-species couplings  $g_\alpha = g$ . In contrast, the temperatures  $\tau_\alpha$  may be different. Thus, the (unrenormalized) dynamic functional is now given by

$$\begin{aligned} \mathcal{J} = \int dt d^d x \lambda \left( \tilde{s}_1 \left( \lambda^{-1} \partial_t + \tau_1 - \nabla^2 + \frac{g}{2} (s_1 - \tilde{s}_1) \right) s_1 \right. \\ \left. + \tilde{s}_2 \left( \lambda^{-1} \partial_t + \tau_2 - \nabla^2 + \frac{g}{2} (s_2 - \tilde{s}_2) \right) s_2 + \frac{1}{2} (\tilde{s}_1 g_{12} + \tilde{s}_2 g_{21}) s_1 s_2 \right) \quad (35) \end{aligned}$$

Note that in the case  $g_{12} = 0$ , the dynamics of species 1 completely decouples from species 2 and vice versa. It follows that  $\Gamma_{\alpha, \alpha\beta} = 0$ , if  $g_{\alpha\beta} = 0$  and  $\alpha \neq \beta$ .

The detailed calculation of the two-loop contributions is presented in Appendices A to C. Adding Eq. (B6) to the zero- and one-loop parts of the selfenergy Eq. (11), we get after renormalization, using Eq. (13) and the scheme Eq. (8),

$$\begin{aligned} \Gamma_{\alpha\alpha} = \lambda \tau_\alpha \left( Z_\tau - \frac{(\mu^2/\tau_\alpha)^{\epsilon/2} u}{2\epsilon(1-\epsilon/2)} \left( 1 + u \left( \frac{2}{\epsilon} - \frac{3}{16} \right) \right) \right) + C_\tau \frac{(\mu^2/\tau_\alpha)^\epsilon u^2}{\epsilon^2} \\ + \lambda q^2 \left( Z_\lambda - \frac{(\mu^2/\tau_\alpha)^{\epsilon/2} u}{8\epsilon} \left( 1 + u \left( \frac{13}{8\epsilon} - \frac{3}{16} \right) \right) \right) + C_{q^2} \frac{(\mu^2/\tau_\alpha)^\epsilon u^2}{\epsilon^2} \\ + i\omega \left( Z_s - \frac{(\mu^2/\tau_\alpha)^{\epsilon/2} u}{4\epsilon} \left( 1 + u \left( \frac{7}{4\epsilon} - \frac{3}{16} \right) \right) \right) + C_\omega \frac{(\mu^2/\tau_\alpha)^\epsilon u^2}{\epsilon^2} \\ + O(\epsilon^0 u^2) \quad (36) \end{aligned}$$

where the constants  $C_i$  are given in Eq. (B7), and  $u = u_\alpha$ . Note that in comparison with Eq. (14) now the one-loop terms acquire renormalizations to  $O(u)$  in order to be consistent in  $O(u^2)$  of the perturbation expansion. These renormalized one-loop terms are needed to compensate non primitive singular terms proportional to  $\ln(\mu^2/\tau)$  emerging now in the  $\epsilon$ -expansion of  $\Gamma_{\alpha\alpha}$ . Of course only primitive UV-divergencies, which means here  $\tau$ -independent ones, have to be regularized by renormalization to

make the theory finite. Eliminating the  $\varepsilon$ -poles from Eq. (36) by the  $Z$ -factors, we find

$$\begin{aligned} Z_s &= 1 + \frac{u}{4\varepsilon} + \frac{u^2}{32\varepsilon} \left( \frac{7}{\varepsilon} - 3 + \frac{9}{2} \ln \frac{4}{3} \right) + O(u^3) \\ Z_\lambda &= 1 + \frac{u}{8\varepsilon} + \frac{u^2}{128\varepsilon} \left( \frac{13}{\varepsilon} - \frac{31}{4} + \frac{35}{2} \ln \frac{4}{3} \right) + O(u^3) \\ Z_\tau &= 1 + \frac{u}{2\varepsilon} + \frac{u^2}{2\varepsilon} \left( \frac{1}{\varepsilon} - \frac{5}{16} \right) + O(u^3) \end{aligned} \quad (37)$$

In the same way as for the selfenergy we proceed with the other vertex functions. Renormalizing Eq. (12) and adding the two-loop contribution Eq. (C13) we get

$$\begin{aligned} \Gamma_{\alpha,\beta} &= G_\varepsilon^{-1/2} \mu^{\varepsilon/2} \lambda u_{\alpha\beta} u^{-1/2} \left( Z_u^{(\alpha\beta)} Z_u^{-1/2} + \frac{(\mu^2/\tau_\alpha)^{\varepsilon/2}}{2\varepsilon} \left( \left( 1 + \frac{2u}{\varepsilon} - \frac{3u}{16} + \frac{u_{\alpha\beta} + u_{\beta\alpha}}{4\varepsilon} \right) u \right. \right. \\ &\quad \left. \left. + \left( 1 + \frac{3u}{2\varepsilon} - \frac{3u}{16} + \frac{u_{\alpha\beta} + u_{\beta\alpha}}{2\varepsilon} \right) \frac{u_{\alpha\beta} + u_{\beta\alpha}}{2} \right) \right. \\ &\quad \left. + \frac{(\mu^2/\tau_\alpha)^\varepsilon}{16\varepsilon} \left( \left( \frac{8}{\varepsilon} + 1 \right) u^2 + \left( \frac{4}{\varepsilon} + \frac{1}{2} \right) uu_{\beta\alpha} + \left( \frac{2}{\varepsilon} + 1 \right) u_{\alpha\beta} u_{\beta\alpha} \right. \right. \\ &\quad \left. \left. + \left( \frac{4}{\varepsilon} + \frac{3}{2} - 3 \ln \frac{4}{3} \right) uu_{\alpha\beta} + \left( \frac{1}{\varepsilon} + \frac{3}{2} \ln \frac{4}{3} \right) (u_{\alpha\beta}^2 + u_{\beta\alpha}^2) \right) \right) + O(\varepsilon^0 u^2) \end{aligned} \quad (38)$$

where  $Z_u = Z_u^{(\alpha\alpha)}$ . Here the  $\varepsilon$ -expansion also shows that the non primitive divergencies  $\sim \ln(\mu^2/\tau)$  cancel. Eventually we find

$$\begin{aligned} Z_u^{(\alpha\beta)} &= 1 + \frac{1}{4\varepsilon} (6u + u_{\alpha\beta} + u_{\beta\alpha}) \\ &\quad + \frac{1}{16\varepsilon} \left( \left( \frac{36}{\varepsilon} - \frac{19}{2} \right) u^2 + \left( \frac{8}{\varepsilon} - \frac{5}{4} \right) uu_{\beta\alpha} + \left( \frac{2}{\varepsilon} - 1 \right) u_{\alpha\beta} u_{\beta\alpha} \right. \\ &\quad \left. + \left( \frac{8}{\varepsilon} - \frac{9}{4} + 3 \ln \frac{4}{3} \right) uu_{\alpha\beta} + \left( \frac{1}{\varepsilon} - \frac{3}{2} \ln \frac{4}{3} \right) (u_{\alpha\beta}^2 + u_{\beta\alpha}^2) \right) + O(u^3) \end{aligned} \quad (39)$$

and in particular

$$Z_u = 1 + \frac{2u}{\varepsilon} + \left( \frac{1}{\varepsilon} - \frac{1}{4} \right) \frac{7u^2}{2\varepsilon} + O(u^3) \quad (40)$$

We are now in the position to calculate the renormalization group functions Eq. (20) from Eqs. (37), (39), and (40) with the help of Eq. (21). They are given by

$$\begin{aligned}\gamma &= -\frac{u}{4} + \left(\frac{2}{3} - \ln \frac{4}{3}\right) \frac{9u^2}{32} + O(u^3) \\ \zeta &= -\frac{u}{8} + \left(\frac{17}{2} - \ln \frac{4}{3}\right) \frac{u^2}{128} + O(u^3) \\ \kappa &= \frac{3u}{8} - \left(\frac{7}{10} + \ln \frac{4}{3}\right) \frac{35u^2}{128} + O(u^3)\end{aligned}\quad (41)$$

and

$$\begin{aligned}\beta_{\alpha\beta} &= \left( -\varepsilon + u + \frac{1}{4}(u_{\alpha\beta} + u_{\beta\alpha}) - \frac{1}{8}u_{\alpha\beta}u_{\beta\alpha} - \frac{3}{16} \ln \frac{4}{3}(u_{\alpha\beta}^2 + u_{\beta\alpha}^2) \right. \\ &\quad \left. - \left(\frac{97}{106} + \ln \frac{4}{3}\right) \frac{53u^2}{64} - \left(\frac{3}{4} - \ln \frac{4}{3}\right) \frac{3uu_{\alpha\beta}}{8} - \frac{5}{32}uu_{\beta\alpha} + O(u^3) \right) u_{\alpha\beta}\end{aligned}\quad (42)$$

Setting  $\beta = \beta_{\alpha\alpha} = 0$ , we find the nontrivial stable fixed point value

$$u_* = \left( 1 + \left(\frac{169}{288} + \frac{53}{144} \ln \frac{4}{3}\right) \varepsilon \right) \frac{2\varepsilon}{3} + O(\varepsilon^2)\quad (43)$$

from which the scaling exponents Eqs. (25) and (27) of the Green functions Eq. (29) follow to second order of the  $\varepsilon$ -expansion as

$$\begin{aligned}\eta &= -\left( 1 + \left(\frac{25}{288} + \frac{161}{144} \ln \frac{4}{3}\right) \varepsilon + O(\varepsilon^2) \right) \frac{\varepsilon}{6} \\ z &= 2 - \left( 1 + \left(\frac{67}{288} + \frac{59}{144} \ln \frac{4}{3}\right) \varepsilon + O(\varepsilon^2) \right) \frac{\varepsilon}{12} \\ \nu &= \frac{1}{2} + \left( 1 + \left(\frac{107}{288} - \frac{17}{144} \ln \frac{4}{3}\right) \varepsilon + O(\varepsilon^2) \right) \frac{\varepsilon}{16}\end{aligned}\quad (44)$$

The first two expansions have been known for a long time from Reggeon field theory<sup>(42, 43)</sup> where a different definition of the exponents is used. The expansion of the exponent  $\nu$  was presented by the current author in ref. 1.

The order parameter exponent  $\beta$ , which enters the scaling law for the mean particle number in the active state  $\langle n \rangle = M \sim |\tau|^\beta$ , follows from Eq. (29) as

$$\beta = \nu \frac{d + \eta}{2} = 1 - \left( 1 - \left( \frac{11}{288} - \frac{53}{144} \ln \frac{4}{3} \right) \varepsilon + O(\varepsilon^2) \right) \frac{\varepsilon}{6} \quad (45)$$

We now turn to the interspecies coupling constants  $u_{\alpha\beta}$  with  $\alpha \neq \beta$ . The fixed point values as the solutions of the equation  $\beta_{\alpha\beta} = 0$ , where  $\beta_{\alpha\beta}$  is the Gell–Mann–Low function Eq. (42), are of three different types:

1. the decoupled fixed point  $u_{12*} = u_{21*} = 0$ , totally unstable for  $u = u_*$ ;
2. the two unidirectional coupled fixed points  $u_{12*} = 0$ ,  $u_{21*} = 2u_* + O(\varepsilon^3)$  and  $u_{21*} = 0$ ,  $u_{12*} = 2u_* + O(\varepsilon^3)$ ;
3. the symmetric fixed point  $u_{12*} = u_{21*} = u_* + O(\varepsilon^3)$ .

To discuss the stability and the crossover between the last two types we try an Ansatz of the form  $u_{\alpha\beta} = w + v\epsilon_{\alpha\beta}$  with  $\epsilon_{12} = -\epsilon_{21} = 1$ . We already know from the one-loop result that  $u_{\alpha\beta}$  is driven by the renormalization flow to the fixed line  $w = u_*$  with a crossover exponent  $\phi_w = \varepsilon/3 + O(\varepsilon^2)$ . Setting therefore  $w = u$  in  $\beta_{\alpha\beta}$  we get

$$\begin{aligned} \beta_{\alpha\beta}(u, v) &= \beta(u) + \varepsilon_{\alpha\beta} \beta_v(u, v) \\ \beta_v(u, v) &= -\left(\frac{1}{8} - \frac{3}{8} \ln \frac{4}{3}\right)(u^2 - v^2) v = -0.017(u^2 - v^2) v \end{aligned} \quad (46)$$

For  $u = u_*$  this equation shows that the symmetric fixed point  $v_* = 0$  is unstable in contrast to the stable unidirectional coupled fixed points  $v_* = \pm u_*$ . The solution of the flow equation  $l d\bar{v}/dl = \beta_v(u_*, \bar{v})$  leads to the crossover

$$\bar{v}(l)^2 = \frac{u_*^2}{1 + (u_*^2/v^2 - 1) l^{\phi_v}} \quad (47)$$

with the very small crossover exponent

$$\phi_v = \left(\frac{1}{4} - \frac{3}{4} \ln \frac{4}{3}\right) u_*^2 = \left(\frac{1}{9} - \frac{1}{3} \ln \frac{4}{3}\right) \varepsilon^2 = 0.0152\varepsilon^2 \quad (48)$$

A qualitative picture of the flow of the interspecies couplings in the plane  $u = u_*$  under renormalization is shown in Fig. 4.

In addition to the four fixed points D (decoupled), S (symmetric), U (unidirectional), the topology of the flow is determined by the symmetry line  $u_{12} = u_{21}$  which acts in the first quadrant as a separatrix between the regions of attraction of the two unidirectional fixed points. There exists another separatrix at the border of these regions, given to first order in  $\varepsilon$  by

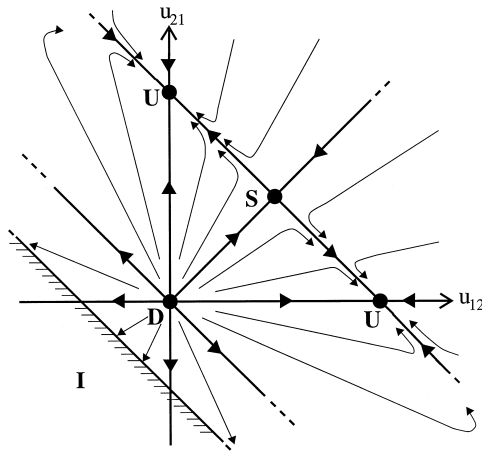


Fig. 4. Flow of the interspecies couplings under renormalization.

$u_{12} + u_{21} = 0$ , where the flow is driven to the hatched line  $u_{12} + u_{21} = -2u_*$ . On the left of this line we have the region of instability I, in which the condition  $\sum_{\alpha, \beta} u_{\alpha\beta} n_\alpha n_\beta \geq 0$  for all positive  $n_\alpha$  is violated. Therefore we conjecture that interspecies couplings with  $u_{12} + u_{21} < 0$  ultimately lead to first order transitions. In Fig. 4 the line of fixed points of the one-loop calculation is also shown. This line is the support of the slow crossover to the unidirectional fixed points which therefore describe the ultimate critical behavior of the MDP universality class.

We conjecture that in the case of colors with different flavors,  $\lambda_\alpha \neq \lambda_\beta$ , all properties are smooth functions of the nonuniversal ratio  $\lambda_\alpha/\lambda_\beta$ , as long as this ratio is sufficiently close to one. Thus, generalizing to different flavors, the topology of the renormalization flow displayed in Fig. 4 should only be smoothly deformed but not destroyed. In particular, the unidirectionally coupled fixed points U should be stable also for  $\lambda_\alpha \neq \lambda_\beta$ .

## VI. SYMMETRIES AND GENERAL FIXED POINT PROPERTIES

Having found the detailed fixed point structure of the model Eq. (35) of two colors with the same flavor in the two-loop approximation, we will now investigate which results are valid to all orders of perturbation theory. According to the considerations of the third chapter, one demonstrates easily that for  $\alpha \neq \beta$  the vertex function  $\Gamma_{\alpha, \alpha\beta} = 0$  if  $g_{\alpha\beta} = 0$ . Thus the lines of unidirectionally coupled models in Fig. 4, namely  $u_{12} = 0$  or  $u_{21} = 0$ , respectively, are invariant under the renormalization flow. Trivially, the decoupled fixed point D is the intersection point of these lines. For  $g_{12} = g_{21}$

the dynamic functional  $\mathcal{J}$  Eq. (35) possesses the symmetry  $s_1 \leftrightarrow s_2$ ,  $\tilde{s}_1 \leftrightarrow \tilde{s}_2$ , and  $\tau_1 \leftrightarrow \tau_2$  from which we find that  $\Gamma_{1,12} = \Gamma_{2,21}$ . Thus, these two vertex functions need the same  $Z$ -factor:  $Z_{12} = Z_{21}$ . It follows that the symmetry line in Fig. 4,  $u_{12} = u_{21}$ , is invariant under renormalization.

Is there a condition that determines the crossover line? The answer is yes. We change to variables corresponding to the total and relative particle numbers, respectively,

$$\begin{aligned} s &= s_1 + s_2, & \tilde{s} &= \frac{1}{2}(\tilde{s}_1 + \tilde{s}_2) \\ c &= s_1 - s_2, & \tilde{c} &= \frac{1}{2}(\tilde{s}_1 - \tilde{s}_2) \end{aligned} \quad (49)$$

Such linear transformations do not alter the measure of the functional integrals, and the dynamic functional changes, in the special case  $\tau_1 = \tau_2 = \tau$ , to

$$\begin{aligned} \mathcal{J} &= \int dt d^d x \lambda \left( \tilde{s} \left( \lambda^{-1} \partial_t + (\tau - \nabla^2) + \left( \frac{g}{4} + \frac{g_{12} + g_{21}}{8} \right) s - \frac{g}{2} \tilde{s} \right) s \right. \\ &\quad + \tilde{c} \left( \lambda^{-1} \partial_t + (\tau - \nabla^2) + \frac{g}{2} s \right) c - \frac{g}{2} \tilde{c}^2 s - g \tilde{s} \tilde{c} c \\ &\quad \left. + \left( \frac{g}{4} - \frac{g_{12} + g_{21}}{8} \right) \tilde{s} c^2 + \frac{g_{12} - g_{21}}{8} \tilde{c} (s^2 - c^2) \right) \end{aligned} \quad (50)$$

We see that in the case  $g_{12} + g_{21} = 2g$  the dynamics of the total particle number decouples from the dynamics of the relative one in the sense that all vertex functions containing  $\tilde{s}$ -, but no  $\tilde{c}$ -legs are zero if  $c$ -legs are attached. In particular  $\Gamma_{\tilde{s},cc} = 0$  and  $\Gamma_{\tilde{s},ss} = -\Gamma_{\tilde{s}\tilde{s},s}$ , which leads to the renormalized interspecies couplings with  $u_{12} + u_{21} = 2u$  and determines the crossover line to all loop-orders. Note that on the symmetry line  $g_{12} = g_{21}$  holds, and the functional  $\mathcal{J}$  is then invariant against  $c \leftrightarrow -c$ ,  $\tilde{c} \leftrightarrow -\tilde{c}$ .

The different fixed point values of the interspecies couplings are now fully determined as the intersection points of the several invariant lines (see Fig. 4): the unidirectional lines  $u_{12} = 0$ ,  $u_{21} = 0$ , the symmetry line  $u_{12} = u_{21}$ , and the crossover line  $u_{12} + u_{21} = 2u$ . Thus, we find to all orders for the symmetric fixed point S:  $u_{12*} = u_{21*} = u_*$  and for the stable unidirectional fixed points U:  $u_{12*} = 0$ ,  $u_{21*} = 2u_*$  and  $u_{21*} = 0$ ,  $u_{12*} = 2u_*$ , respectively. These are of course the results that we have found explicitly in the two-loop approximation.

The unidirectionally coupled model, which describes the generic scaling properties of the MDP processes, exhibits another symmetry. Using

$g_{12}=0$  and  $g_{21}=2g'$ , we write the dynamic functional of this model in the form

$$\begin{aligned} \mathcal{J}_u = \int dt d^d x \lambda \left( \tilde{s}_1 \left( \lambda^{-1} \partial_t + (\tau_1 - \nabla^2) + \frac{g}{2} (s_1 - \tilde{s}_1) \right) s_1 \right. \\ \left. + \tilde{s}_2 \left( \lambda^{-1} \partial_t + (\tau_2 - \nabla^2) + \frac{g}{2} (s_2 - \tilde{s}_2) \right) s_2 + \tilde{s}_2 (-\sigma + g' s_2) s_1 \right) \quad (51) \end{aligned}$$

Here we have introduced a further harmonic unidirectional coupling  $\sim \sigma$ , which corresponds to an additional linear term:  $\partial n_2 / \partial t = \dots + \lambda \sigma n_1$  in the Langevin equation for species 2. This term was first considered by Täuber *et al.*<sup>(30)</sup> in their study of the nonequilibrium critical behavior in unidirectionally coupled DP processes. Of course,  $\sigma$  is a relevant parameter like the temperatures  $\tau_\alpha$  and needs its own multiplicative renormalization with a factor  $Z_\sigma$  determined by

$$\sigma \rightarrow \hat{\sigma} = Z_\sigma Z_\lambda^{-1} \sigma \quad (52)$$

in such a way that the renormalized vertex function with an insertion  $\Gamma_{2,1;(\tilde{s}_2 s_1)}$  is finite. The settings  $\sigma = \tau_1 = \tau_2 = 0$  define the multicritical point.

We generalize time reflection (rapidity reversal)  $s(t) \leftrightarrow -\tilde{s}(-t)$ , which is broken by the DP transition to an active state, to read

$$\begin{aligned} s_1(t) &\rightarrow -\tilde{s}_2(-t), & \tilde{s}_1(t) &\rightarrow -s_2(-t) - s_1(-t) \\ \tilde{s}_2(t) &\rightarrow -s_1(-t), & s_2(t) &\rightarrow -\tilde{s}_1(-t) + \tilde{s}_2(-t) \end{aligned} \quad (53)$$

Under this transformation the functional  $\mathcal{J}_u$  changes to

$$\begin{aligned} \mathcal{J}_u \rightarrow \int dt d^d x \lambda \left( \tilde{s}_1 \left( \lambda^{-1} \partial_t + (\tau_2 - \nabla^2) + \frac{g}{2} (s_1 - \tilde{s}_1) \right) s_1 \right. \\ \left. + \tilde{s}_2 \left( \lambda^{-1} \partial_t + (\tau_1 - \nabla^2) + \frac{g}{2} (s_2 - \tilde{s}_2) \right) s_2 \right. \\ \left. + \tilde{s}_2 (-\sigma + \tau_1 - \tau_2 + g' s_2) s_1 + (g - g') (\tilde{s}_1 \tilde{s}_2 s_1 - \tilde{s}_2^2 s_1 + \tilde{s}_2 s_1 s_2) \right) \quad (54) \end{aligned}$$

We learn from this relation that in the case  $g = g'$ ,  $\mathcal{J}_u$  gains a higher symmetry at the multicritical point, which is not destroyed by renormalization. It follows in this case that  $\sigma$  renormalizes in the same way as the  $\tau_\alpha$ , which implies  $Z_\sigma = Z_\tau$ , and that the renormalized couplings are related by  $u'_* = u_{21*} = u_*$  at the fixed point, as we already know from above. From

these considerations follows that the crossover exponent, which is defined by the scaling invariants  $\sigma/\tau_\alpha^\Phi$ , is given by  $\Phi = 1$ .

## VII. THE THHG MODEL

Recently Täuber *et al.*<sup>(30)</sup> (in the following abbreviated by THHG) introduced a general unidirectionally coupled DP process with species-independent diffusion coefficients. This model was motivated by a study of Alon *et al.*<sup>(28)</sup> on a nonequilibrium growth model for adsorption and desorption of particles which displays a roughening transition. At first we will complete the THHG model by a cross diffusion term that is generated by coarse graining and is indispensable for a full renormalization. Introducing a forminvariance transformation we derive the one-loop renormalization group functions for the invariant combination of the parameters. Then we show that the fixed point of the THHG model coincides with the unidirectional model of the last section. The line of fixed points found by THHG arises from the application of the forminvariance transformation. Lastly we demonstrate that the linear coupling between the two species renormalizes in the same way as the temperature variables. Thus the crossover exponent of this linear interspecies coupling is simply given by  $\Phi = 1$ .

The THHG-model takes in our dynamic functional language the form

$$\begin{aligned} \mathcal{J}_{THHG} = & \int dt d^d x \lambda \left( \tilde{s}_1 \left( \lambda^{-1} \partial_t + \tau_1 - \nabla^2 + \frac{g}{2} (s_1 - \tilde{s}_1) \right) s_1 \right. \\ & + \tilde{s}_2 \left( \lambda^{-1} \partial_t + \tau_2 - \nabla^2 + \frac{g}{2} (s_2 - \tilde{s}_2) \right) s_2 \\ & \left. - \tilde{s}_2 \left( \sigma + b \nabla^2 - \frac{f_1}{2} s_1 - f_2 s_2 + f'_2 \tilde{s}_1 + \frac{f'_1}{2} \tilde{s}_2 \right) s_1 \right) \quad (55) \end{aligned}$$

In contrast to THHG, we have introduced an additional cross diffusion term  $\sim \tilde{s}_2 \nabla^2 s_1$  that is indispensable for a complete renormalization of the general model. In physical terms, coarse graining will always produce cross diffusion in this coupled model. Strictly, however, coarse graining does even more: it also produces a term proportional to the time derivative of the density of the first species in the Langevin equation of the second one—besides further irrelevant couplings. Accordingly, in the functional  $\mathcal{J}_{THHG}$  a term proportional to  $\tilde{s}_2 \partial_t s_1$  arises. However, such a term can be eliminated by a suitable redefinition of the fields

$$\tilde{s}_1 \rightarrow \tilde{s}_1 - \beta \tilde{s}_2, \quad s_1 \rightarrow s_1, \quad \tilde{s}_2 \rightarrow \tilde{s}_2, \quad s_2 \rightarrow s_2 - \beta s_1 \quad (56)$$



so that the harmonic parts with the time derivatives in the dynamic functional remain diagonal. Note that in the special case  $b=f_1=f'_1=f'_2=0$ ,  $f_2=g'$  the functional  $\mathcal{J}_{THHG}$  Eq. (55) is identical to  $\mathcal{J}_u$ , Eq. (51), from which we know that it is fully renormalizable, in particular, in the case  $g'=g$ , by the  $Z$ -factors Eqs. (37) and (40). In the following we will demonstrate that the asymptotic properties of the THHG-model indeed belong to the universality class described by  $\mathcal{J}_u$ .

It is well known<sup>(44)</sup> that the infinitesimal generators of a continuous transformation of the fundamental fields, which leads to a form invariance of the describing statistical functional, define redundant operators. These redundant operators unnecessarily contaminate the renormalization group. Therefore they should be avoided from the outset. Here we introduce a linear, homogeneous transformation between the fields, which does not change the form of the functional  $\mathcal{J}_{THHG}$  Eq. (55). Let us call it the  $\alpha$ -transformation:

$$\tilde{s}_1 \rightarrow \tilde{s}_1 - \alpha \tilde{s}_2, \quad s_1 \rightarrow s_1, \quad \tilde{s}_2 \rightarrow \tilde{s}_2, \quad s_2 \rightarrow s_2 + \alpha s_1 \quad (57)$$

Note the difference to the  $\beta$ -transformation Eq. (56), which is exploited to eliminate a  $\tilde{s}_2 \partial_t s_1$ -term from  $\mathcal{J}_{THHG}$ . This  $\alpha$ -transformation leads to new coupling constants that we denote with a bar:

$$\begin{aligned} \bar{\sigma} &= \sigma + \alpha(\tau_1 - \tau_2) \\ \bar{f}_1 &= f_1 + 2\alpha f_2 + \alpha(\alpha - 1)g, & \bar{f}_2 &= f_2 + \alpha g \\ \bar{f}'_1 &= f'_1 - 2\alpha f'_2 + \alpha(\alpha + 1)g, & \bar{f}'_2 &= f'_2 - \alpha g \end{aligned} \quad (58)$$

Now we renormalize the THHG-model by the scheme

$$\begin{aligned} \tilde{s}_1 &\rightarrow \hat{\tilde{s}}_1 = Z_s^{1/2}(\tilde{s}_1 + A\tilde{s}_2), & s_1 &\rightarrow \hat{s}_1 = Z_s^{1/2}s_1 \\ s_2 &\rightarrow \hat{s}_2 = Z_s^{1/2}(s_2 + As_1), & \tilde{s}_2 &\rightarrow \hat{\tilde{s}}_2 = Z_s^{1/2}\tilde{s}_2 \\ \tau_\alpha &\rightarrow \hat{\tau}_\alpha = Z_\lambda^{-1}Z_\tau\tau_\alpha, & g &\rightarrow \hat{g} = Z_s^{-1/2}Z_\lambda^{-1}Z_g g \\ \sigma &\rightarrow \hat{\sigma} = Z_\lambda^{-1}(Z_\sigma\sigma + (Y_1 + AZ_\tau)\tau_1 + (Y_2 + AZ_\tau)\tau_2) \\ b &\rightarrow \hat{b} = Z_\lambda^{-1}Z_b b - 2A \\ f_1 &\rightarrow \hat{f}_1 = Z_s^{-1/2}Z_\lambda^{-1}(Z_1 f_1 - A(1-A)Z_g g - 2AZ_2 f_2) \\ f'_1 &\rightarrow \hat{f}'_1 = Z_s^{-1/2}Z_\lambda^{-1}(Z'_1 f'_1 - A(1-A)Z_g g - 2AZ'_2 f'_2) \\ f_2 &\rightarrow \hat{f}_2 = Z_s^{-1/2}Z_\lambda^{-1}(Z_2 f_2 - AZ_g g) \\ f'_2 &\rightarrow \hat{f}'_2 = Z_s^{-1/2}Z_\lambda^{-1}(Z'_2 f'_2 - AZ_g g) \end{aligned} \quad (59)$$

where  $g=(u\mu^\varepsilon/G_\varepsilon)^{1/2}$ ,  $Z_g=Z_u^{1/2}$ , and the  $Z_i$  with  $i=s, \lambda, \tau, u$  are given by Eqs. (37) and (40). Besides  $u$ , dimensionless coupling constants are defined as  $v_\alpha^{(l)}=(G_\varepsilon/\mu^\varepsilon)^{1/2} f_\alpha^{(l)}$ . The scheme Eq. (59) is chosen in such a way that the renormalized dynamic functional reads

$$\begin{aligned} \mathcal{J}_{THHG} = & \int dt d^d x \lambda \left( \tilde{s}_1 \left( Z_s \lambda^{-1} \partial_t + Z_\tau \tau_1 - Z_\lambda \nabla^2 + \frac{Z_g g}{2} (s_1 - \tilde{s}_1) \right) s_1 \right. \\ & + \tilde{s}_2 \left( Z_s \lambda^{-1} \partial_t + Z_\tau \tau_2 - Z_\lambda \nabla^2 + \frac{Z_g g}{2} (s_2 - \tilde{s}_2) \right) s_2 \\ & + \tilde{s}_2 \left( 2AZ_s \lambda^{-1} \partial_t - Z_\sigma \sigma - Y_1 \tau_1 - Y_2 \tau_2 - Z_b b \nabla^2 \right. \\ & \left. \left. + \frac{Z_1 f_1}{2} s_1 + Z_2 f_2 s_2 - Z_2' f_2' \tilde{s}_1 - \frac{Z_1' f_1'}{2} \tilde{s}_2 \right) s_1 \right) \end{aligned} \quad (60)$$

Note that the counter term  $2AZ_s$  serves to cancel primitive divergencies arising in the vertex function  $\partial_\omega \Gamma_{\tilde{s}_2 s_1}$ . In dimensional regularization and minimal renormalization the counterterms  $A$  and  $Y_\alpha$  are given by series in  $\varepsilon^{-1}$  beginning with simple poles  $\sim A^{(1)}/\varepsilon$  and  $\sim Y_\alpha^{(1)}/\varepsilon$ , respectively. We have formerly shown that only the residua of these poles determine the renormalization group functions.

It is now appropriate to define  $\alpha$ -transformation invariant dimensionless coupling constants as

$$w_1 = \sqrt{u} (v_1 + v_2) - v_2^2, \quad w_1' = \sqrt{u} (v_1' + v_2') - v_2'^2, \quad w_2 = \sqrt{u} (v_2 + v_2') \quad (61)$$

The somewhat lengthy but simple calculation of all the one-loop renormalizations leads to the Gell–Mann–Low functions of the renormalization group equation. In part, in the case  $b=0$ , they can be derived from the results of THHG.<sup>(30)</sup> If we set  $u$  to its fixed point value  $u_*$  and define as usual  $\beta_p = \partial p / \partial \ln \mu|_0$ , where  $p$  is any of the coupling constants, we obtain for the  $\beta$ -functions of the invariant couplings

$$\begin{aligned} 8\beta_{w_1} &= 4(2w_1 + w_2 - u_*) w_2 + 4u_* w_1' + 16u_* a - u_* b(8w_1 + 10w_2) + 9(u_* b)^2 \\ 8\beta_{w_1'} &= 4(2w_1' + w_2 - u_*) w_2 + 4u_* w_1 + 16u_* a - u_* b(8w_1' + 10w_2) + 9(u_* b)^2 \\ 4\beta_{w_2} &= 8(w_2 - u_*) w_2 + 4u_* (w_1 + w_1') + 8u_* a - 6u_* b w_2 + 3(u_* b)^2 \\ 8u_* \beta_b &= u_* (w_1 + w_1') + (w_2 - u_*) w_2 + 16u_* a - u_* b(w_2 + u_*) + (u_* b)^2 \end{aligned} \quad (62)$$

The function  $a$  results from the additive renormalization  $A$  and mixes the renormalized fields in a correlation or response function under application of the renormalization group as

$$\left[ \mathcal{D}_\mu + \frac{\gamma}{2} \right] \{ \tilde{s}_1, s_1, \tilde{s}_2, s_2 \} = \{ -a\tilde{s}_2, 0, 0, -as_1 \} \quad (63)$$

Here  $\mathcal{D}_\mu$  is the renormalization group differential operator now given by

$$\mathcal{D}_\mu = \mu \partial_\mu + \zeta \lambda \partial_\lambda + \kappa_\tau (\tau_1 \partial_{\tau_1} + \tau_2 \partial_{\tau_2}) + (\kappa_\sigma \sigma + \kappa_1 \tau_1 + \kappa_2 \tau_2) \partial_\sigma + \sum_p \beta_p \partial_p \quad (64)$$

with  $\kappa_\sigma = \gamma_\lambda - \gamma_\sigma$ ,  $\kappa_\alpha = -y_\alpha - a$ , and Eq. (63) acts on Green functions. The  $y_\alpha = -\sum_p \partial_p Y_\alpha^{(1)}$  result from the additive renormalizations  $Y_\alpha$ . The function  $a = -\sum_p \partial_p A^{(1)}$  is found to be

$$a = -\frac{1}{16} \left( 2w_1 + w'_1 + 2 \left( \frac{w_2}{u} - 1 \right) w_2 - 4bw_2 + 3ub^2 \right) \quad (65)$$

The new renormalizations yield

$$\begin{aligned} \gamma_\sigma &= \frac{1}{2} (bu - w_2) \\ y_1 &= \frac{1}{2} \left( \sqrt{u} v_1 + v_2 v'_2 - \frac{3b}{2} \sqrt{u} v_2 - \frac{b}{2} \sqrt{u} v'_2 + \frac{3b^2}{4} u \right) \\ y_2 &= \frac{1}{2} \left( \sqrt{u} v'_1 + v_2 v'_2 - \frac{3b}{2} \sqrt{u} v'_2 - \frac{b}{2} \sqrt{u} v_2 + \frac{3b^2}{4} u \right) \end{aligned} \quad (66)$$

In order to determine the fixed-point solutions of Eq. (62),  $\beta_{p*} = 0$ , we impose the condition  $a_* = b_* = 0$ . This yields  $w_{1*} = w'_{1*} = 0$ , with  $w_{2*} = 0$  (unstable) or  $w_{2*} = u_*$  (stable). It can easily be checked that these solutions are consistent with the full set of Eqs. (62) and (65). These are the solutions found by THHG.<sup>(30)</sup> Note that on the line of fixed points generated from the stable fixed point by the  $\alpha$ -transformation a minimally coupled fixed point with  $v_{1*} = v'_{1*} = v'_{2*} = 0$ ,  $v_{2*} = \sqrt{u_*}$  is found.

Now one has to prove stability of the fixed points of the full equations (62) without using the constraints  $a_* = b_* = 0$ . A linearization about  $w_1 = w'_1 = b = 0$  and either  $w_2 = 0$  or  $w_2 = u_*$  shows that the flow of  $b$  is unstable for  $w_2 = 0$ , whereas it shows full stability of the fixed point for  $w_2 = u_*$ . This vindicates the neglect of  $a, b$  and the corresponding counterterms  $A$  and  $Z_b b$  in ref. 30 but only at the stable line of fixed points, which is

generated from the fixed point by the  $\alpha$ -transformation. However, without further knowledge this statement is only correct in the one-loop calculation and could be violated in higher loop orders. We will show that the stable fixed point is given by  $w_{1*} = w'_{1*} = 0$ ,  $w_{2*} = u_*$  to all orders of the loop expansion. As a consequence, the fixed point values  $a_*$ ,  $b_*$ ,  $y_{1*}$ ,  $y_{2*}$  are zero, and  $\gamma_{\sigma*} = \gamma_{\tau*}$ . This leads to a crossover exponent  $\Phi = 1$  where  $\Phi$  determines the scaling of  $\sigma/|\tau_i|^\Phi$ .

Indeed, we see from Eq. (61) that the stable one-loop order fixed point belongs, up to an  $\alpha$ -transformation, to the dynamic functional  $\mathcal{J}_{THHG}$ , Eq. (60), with coupling constants  $f_1 = f'_1 = f'_2 = 0$  and  $f_2 = g$ , i.e., model  $\mathcal{J}_u$ , Eq. (55), with the additional constraint  $g' = g$ . Above it was shown that this equality leads to the time reflexion Eq. (53) as a higher symmetry. This higher symmetry is preserved under renormalization and, because  $\mathcal{J}_u$  is fully renormalizable, we have the result  $w_{1*} = w'_{1*} = 0$ ,  $w_{2*} = u_*$  and  $\Phi = 1$  to all loop orders.

Computations based on the dynamic functional  $\mathcal{J}_u$ , Eq. (55), are much easier to perform than calculations using the complete model  $\mathcal{J}_{THHG}$ , Eq. (60). Thus it may be possible to find the equation of state for  $M_2 = g\langle s_2 \rangle$  to second order and check the assumptions made in ref. 30 on the reexponentiation of logarithms to yield the new order parameter exponent  $\beta_2$  of that paper (for a calculation of the equation of state for  $M_1 = g\langle s_1 \rangle$  to two-loop order see ref. 18).

The model  $\mathcal{J}_u$  describes the coupled DP processes near the multicritical point  $\tau_1 = \tau_2 = \sigma = 0$ . What is needed for a thorough calculation of  $\beta_2$  is a theory that comprises the limit  $\sigma \rightarrow \infty$ . Therefore our considerations here do not solve the problem addressed in ref. 30, namely the determination of the scaling exponent  $\beta_2$  that controls the scaling  $M_2 \propto \sigma^{\beta_1 - \beta_2} (\tau_{1c} - \tau_1)^{\beta_2}$  where species 1 is in its active phase. THHG calculate  $\beta_2$  by reexponentiation of logarithms. (We have done a recalculation and find a slightly different value  $\beta_2 = 1/2 - 13\varepsilon/96 + O(\varepsilon^2)$ . The difference arises from a subleading term resulting from the ominous peculiar diagram Fig. 9(c) in ref. 30.) However, the approach of THHG relies on the assumption that simple reexponentiation is possible. To derive such scaling properties faithfully, one indeed has to solve the crossover problem  $\sigma \rightarrow \infty$ .

## VIII. TOURNAMENTS

As we have shown in the previous chapters, the scaling behavior of MDP near the absorbing state transition coincides with that of ordinary one-species DP. In particular, the critical exponents are the same. An important consequence is, that the critical behavior is independent of the

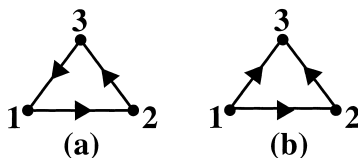


Fig. 5. Cyclic and hierarchic tournament of three species.

number of colors. The characteristic asymptotic feature of MDP manifests itself in the asymptotic unidirectional coupling of each pair of colors. This unidirectional behavior of the couplings of an interacting population is summarized in the following graphical picture. Consider a graph where each node represents one color. The stable fixed points are then represented by the so-called tournaments, that are the complete graphs with directed edges.<sup>(46)</sup> The directed edge from color  $\alpha$  to color  $\beta$  stands for the influence of  $\alpha$  on  $\beta$  in the respective equations of motion. In particular, for a population of three species there exist two different tournaments (up to permutation of the colors) which we call “cyclic” and “hierarchic,” Fig. 5.

One gets a graphical picture for the not fully stable fixed points either by deleting the directionality of the edges between the nodes for symmetric couplings of the corresponding colors, or by completely deleting the edges for the uncoupled pairs (uncomplete tournaments).

To get a simple qualitative impression of the behavior of the dynamic system of a population corresponding to a tournament of colors with the same flavor, we consider, in the active region with all  $r_\alpha := -\tau_\alpha/2g \geq 0$ , a renormalized mean-field theory of the equations (1), (2) for spatially homogeneous densities  $\bar{n}_a$  and set  $g_{\alpha\beta} = g(1 + \varepsilon_{\alpha\beta})$  with  $\varepsilon_{\alpha\alpha} = 0$ . We redefine the time scale  $t \rightarrow 2t/\lambda g$  and get

$$\partial_t \bar{n}_a = \left( r_\alpha - \sum_\beta (1 + \varepsilon_{\alpha\beta}) \bar{n}_\beta \right) \bar{n}_a \quad (67)$$

Then the decision, which of all the  $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$  is equal  $\pm 1$ , defines the tournament. The edge between a pair of species is directed from  $\beta$  to  $\alpha$  if  $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha} = +1$  and vice versa. The directions of the edges therefore represent the unidirectional “pressure” on the reproduction rate resulting from one color to another. Despite the simplicity of the Eq.(67) they can generate a complex dynamic behavior (see e.g., refs. 26 and 27). First, let us consider the stationary states of Eq. (67) for a population consisting of three species. In the ternary phase diagram spanned by the positive rates  $r_1, r_2, r_3$  in the subspace  $\sum_\alpha r_\alpha = r = \text{const.}$ , one finds different regions with one, two or all three species alive, Fig. 6. These regions are bounded by

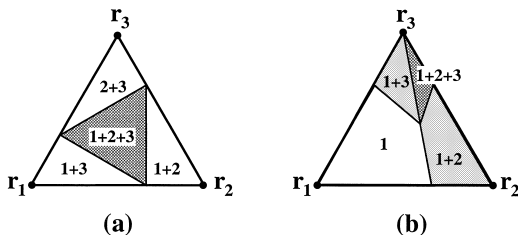


Fig. 6. Phase diagrams of the cyclic and hierarchic tournament.

critical lines with absorbing state transitions where some colors become extinct. The dynamic behavior of the hierarchic tournament Fig. 5(b) is relatively simple: from each nonequilibrium initial state  $\{\bar{n}_\alpha^{(0)}\}$ , the system relaxes to the stationary state which is a stable node. In the case of the cyclic tournament Fig. 5(a), for rates  $\{r_\alpha\}$  such that we have a three species stationary state, one also finds regions for which the ultimate relaxation behavior is characterized by attracting nodes. However, for rates which lead to stationary states belonging to the crosshatched area in the ternary diagram spanned by the  $\{\bar{n}_\alpha\}$ , Fig. 7, we find stable spirals (damped cyclic relaxation) as the ultimate relaxation behavior (qualitatively pictured by the trajectory in Fig. 7).

For stationary points near the middle of Fig. 7 the damping is very small and cyclic behavior dominates the dynamics. Especially in the middle of the diagram, i.e., for equal rates  $r_1=r_2=r_3=r/3$ , the motion is not damped anymore. In this case, the dynamic system Eq. (67) is known as a special form of the May–Leonard model<sup>(47)</sup> that has been extensively studied in mathematical biology.<sup>(26, 48–50)</sup> Finally, after a relaxation in the plane  $\bar{n}=\sum_\alpha \bar{n}_\alpha=r$ , there exists another constant of motion  $m=\prod_\alpha \bar{n}_\alpha$  and the dynamic behavior is characterized by limit cycles around the neutral stationary point  $\{\bar{n}_\alpha^{(0)}=r/3\}$ .

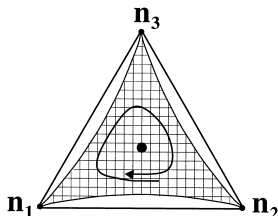


Fig. 7. Dynamic behavior of the cyclic tournament.

## IX. EPILOGUE

In this paper, we have studied stochastic multispecies models of populations that evolve near the extinction threshold of all the species and therefore have many absorbing states. The critical properties of these multicolored directed percolation processes (MDP) at their multicritical points and at all appending lines of continuous transitions with extinction of some species are governed by the well known DP exponents. This findings seem to agree with simulational results on a two species system.<sup>(51)</sup> In other regions of the phase diagram of course more complicated critical behavior may arise such as multicritical points with different scaling exponents.<sup>(52)</sup> The models considered here have many absorbing states (each combination of colors may go extinct irrespective of the other ones). Therefore they are different from the models considered by Grinstein *et al.*<sup>(14)</sup> These authors showed that multispecies systems with one absorbing state belong to the DP universality class.

As a key result we have shown that the characteristic asymptotic feature of the interspecies correlations and the phase diagram of MDP shows up in the asymptotic unidirectional coupling of each pair of species which breaks color-symmetry. A special result of our analysis is the very slow crossover to this asymptotic unidirectionality which may be seen in computer simulations. This slow crossover eventually leads to a system working cooperatively. It is interesting that the asymmetry between the species well described by tournaments seems to be the condition for this cooperation near extinction. The model considered in this paper is a simple but universal model of a cooperative society. Therefore, it should have many applications in diverse fields of natural and even social science. Competition and extinction are of course subjects heavily studied in theoretical biology. Compared to models as covered e.g., in the monograph of Hofbauer and Sigmund,<sup>(26)</sup> our work features a more realistic local description of the interactions between the species, their diffusional motion, and it includes local fluctuations. The features are encoded in the form of local stochastic partial differential equations. Accompanied by coarse graining and renormalization, this gives an universal macroscopic picture of cooperativity near the critical states of extinction.

## APPENDIX A: TWO-LOOP INTEGRALS

In the calculation we encounter momentum integrals of the type

$$I_{kl;m} = G_\varepsilon^{-2} \tau^\varepsilon \int_{q_1, q_2} \frac{1}{(q_1^2 + \tau)^k (q_2^2 + \tau)^l (q_1^2 + q_2^2 + (q_1 + q_2)^2 + 3\tau)^m} \quad (\text{A1})$$

where  $G_\varepsilon = \Gamma(1 + \varepsilon/2)/(4\pi)^{d/2}$ ,  $\varepsilon = 4 - d$ , and  $\int_{\mathbf{q}} \cdots = (2\pi)^{-d} \int d^d q \cdots$ . They can be derived from two ‘‘mother’’ integrals

$$M^{(1)}(a, b; c) = G_\varepsilon^{-2} \int_{q_1, q_2} \frac{1}{(q_1^2 + a)(q_2^2 + b)(q_1^2 + q_2^2 + (q_1 + q_2)^2 + c)} \quad (\text{A2})$$

$$M^{(2)}(a; c) = G_\varepsilon^{-2} \int_{q_1, q_2} \frac{1}{(q_1^2 + a)(q_1^2 + q_2^2 + (q_1 + q_2)^2 + c)}$$

by taking derivatives with respect to the parameters  $a, b, c$ . Discarding nonsingular terms, we find in dimensional regularization

$$M^{(1)}(a, b; c) = -\frac{1}{\varepsilon} \left( \frac{a^{1-\varepsilon} + b^{1-\varepsilon}}{\varepsilon} + \frac{3(a+b)}{2} \left( 1 - \ln \frac{4}{3} \right) + c \ln \frac{4}{3} \right)$$

$$M^{(2)}(a; c) = \frac{1}{4\varepsilon} \left( \frac{2c - 3a}{\varepsilon} a^{1-\varepsilon} + ac \left( 1 + \ln \frac{4}{3} \right) - 3a^2 \left( 1 + \frac{1}{2} \ln \frac{4}{3} \right) + \frac{c^2}{3} \right) \quad (\text{A3})$$

These formulas yield the singular parts of the integrals (A1) as

$$I_{11;1}^{(SP)} = -\frac{1}{\varepsilon^2} (2 + 3\varepsilon), \quad I_{11;2}^{(SP)} = \frac{1}{\varepsilon} \ln \frac{4}{3}$$

$$I_{12;1}^{(SP)} = \frac{1}{2\varepsilon^2} \left( 2 + \varepsilon - 3\varepsilon \ln \frac{4}{3} \right), \quad I_{10;3}^{(SP)} = \frac{1}{12\varepsilon}, \quad I_{20;1}^{(SP)} = \frac{3}{2\varepsilon} \quad (\text{A4})$$

$$I_{20;2}^{(SP)} = \frac{1}{4\varepsilon^2} \left( 2 - \varepsilon + \varepsilon \ln \frac{4}{3} \right), \quad I_{30;1}^{(SP)} = -\frac{3}{8\varepsilon^2} \left( 2 + \varepsilon + \varepsilon \ln \frac{4}{3} \right)$$

## APPENDIX B: TWO-LOOP SELFENERGY DIAGRAMS

In Fig. 8 the two-loop selfenergy diagrams are drawn. Diagram Fig. (8a) leads to

$$4(a) = \frac{(\lambda g)^4}{2} \int_{q_1, q_2} \int_{i=1}^3 dt_i e^{-i\omega t_i} G(\mathbf{q}/2 + \mathbf{q}_1, t_1) G(\mathbf{q}/2 - \mathbf{q}_1, t_1 - t_2)$$

$$\times G(\mathbf{q}/2 - \mathbf{q}_1, t_3) G(\mathbf{q}/4 + \mathbf{q}_2, t_1 - t_3) G(\mathbf{q}/4 - \mathbf{q}_1 - \mathbf{q}_2, t_1 - t_3) \quad (\text{B1})$$

where  $G(\mathbf{q}, t)$  is the propagator (Eq. (7)) and we always set  $\tau_\alpha = \tau > 0$  as an IR regulator. Noting that  $e^{-i\omega t_1} G(t_1) = (e^{-i\omega(t_1-t_2)} G(t_1 - t_2))(e^{-i\omega(t_2-t_3)} G(t_2 - t_3)) \times (e^{-i\omega t_3} G(t_3))$  factorizes in the integral, the time integrations over the



intervals  $(t_1 - t_2)$ ,  $(t_2 - t_3)$ ,  $t_3$  are easily performed. An expansion in  $i\omega$  and  $q^2$  to linear order eventually yields

$$4(a) = \frac{\lambda g^4}{8} G_\varepsilon^2 \tau^{-\varepsilon} \left( \tau I_{20;1} - \frac{i\omega}{\lambda} (I_{30;1} + I_{20;2}) - \frac{q^2}{2} \left( I_{30;1} + \frac{3}{4} I_{20;2} + \frac{1}{2d} (I_{20;3} - I_{10;3}) \right) \right) \quad (\text{B2})$$

where the  $I_{kl;m}$  denote the integrals defined in Eq. (A1). Extracting the singular parts using Eq. (A4), we obtain

$$4(a) = \frac{\lambda g^4}{32\varepsilon} G_\varepsilon^2 \tau^{-\varepsilon} \left( 6\tau + \left( \frac{2}{\varepsilon} + 5 + \ln \frac{4}{3} \right) \frac{i\omega}{2\lambda} + \left( \frac{3}{\varepsilon} + \frac{55}{12} + \frac{3}{2} \ln \frac{4}{3} \right) \frac{q^2}{4} \right) \quad (\text{B3})$$

In the same way we calculate the second selfenergy diagram Fig. 8(b):

$$4(b) = (\lambda g)^4 \int_{\mathbf{q}_1, \mathbf{q}_2} \int_{t_1}^3 \prod_{i=1}^3 dt_i e^{-i\omega t_i} G(\mathbf{q}/2 + \mathbf{q}_1, t_1 - t_2) G(\mathbf{q}/2 - \mathbf{q}_1, t_1 - t_3) \\ \times G(\mathbf{q}_1 - \mathbf{q}_2, t_2 - t_3) G(\mathbf{q}/2 + \mathbf{q}_2, t_2) G(\mathbf{q}/2 - \mathbf{q}_2, t_3) \\ = \frac{\lambda g^4}{4} G_\varepsilon^2 \tau^{-\varepsilon} \left( \tau I_{11;1} - \frac{i\omega}{\lambda} (I_{12;1} + I_{11;2}) - \frac{q^2}{2} \left( I_{12;1} + I_{11;2} + \frac{2}{d} (I_{11;3} + 2I_{10;3} - I_{11;2}) \right) \right) \quad (\text{B4})$$

up to higher orders in  $i\omega$  and  $q^2$ . Extracting again the singular parts, we find

$$4(b) = -\frac{\lambda g^4}{2\varepsilon} G_\varepsilon^2 \tau^{-\varepsilon} \left( \left( \frac{1}{\varepsilon} + \frac{3}{2} \right) \tau + \left( \frac{2}{\varepsilon} + 1 - \ln \frac{4}{3} \right) \frac{i\omega}{4\lambda} + \left( \frac{1}{\varepsilon} + \frac{7}{12} - \ln \frac{4}{3} \right) \frac{q^2}{4} \right) \quad (\text{B5})$$

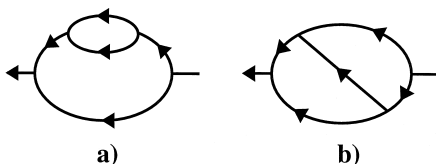


Fig. 8. Two-loop selfenergy diagrams.

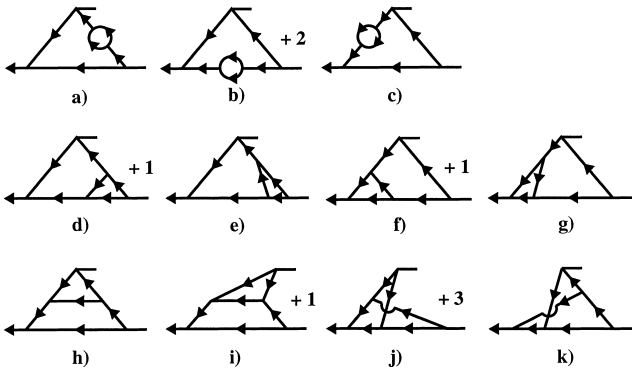


Fig. 9. Two-loop vertex diagrams; the numbers show the other time orderings.

Summing up, we finally get from 4(a) and (4b) the two-loop contribution of the selfenergy as

$$\Gamma_{\alpha\alpha}^{(2-loop)} = \frac{\lambda g^4 G_\epsilon^2}{\epsilon^2} \tau^{-\epsilon} \left( C_\tau \tau + C_\omega \frac{i\omega}{\lambda} + C_{q^2} q^2 \right) \tag{B6}$$

where

$$C_\tau = \frac{1}{2} + \frac{9\epsilon}{16}, \quad C_\omega = \frac{7}{32} \left( 1 + \frac{3\epsilon}{14} - \frac{9\epsilon}{14} \ln \frac{4}{3} \right), \tag{B7}$$

$$C_{q^2} = \frac{13}{128} \left( 1 + \frac{19\epsilon}{52} - \frac{35\epsilon}{26} \ln \frac{4}{3} \right)$$

### APPENDIX C: TWO-LOOP VERTEX DIAGRAMS

Figure 9 presents the eleven two-loop vertex diagrams. They can be calculated by the same method as the selfenergy diagrams. Here the external frequencies and momenta can be set to zero because the expansion in these variables does not lead to primitive divergencies. As an example we show the calculation of the diagram Fig. 9(i) explicitly.

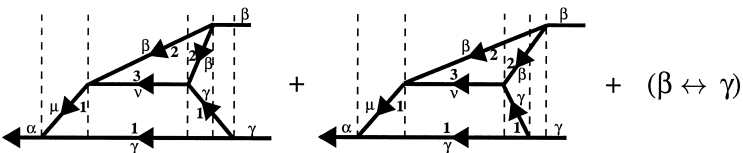


Fig. 10. Two different time orderings of a vertex diagram.

The two possible different time orderings of the vertices are shown in Fig. 10.

The symmetry factor of the diagrams is one, but one also has to add the two diagrams arising from the interchange of the color indices  $\beta$  and  $\gamma$ . After the factorization of the propagators, the integration over the time intervals, indicated by the broken lines, is trivial and leads to the expression (with the abbreviations  $\kappa_i = \lambda(\tau + q_i^2)$ ,  $\mathbf{q}_3 = \mathbf{q}_1 + \mathbf{q}_2$ )

$$\begin{aligned}
 5(i) &= -\frac{\lambda^5}{8} \sum_{\mu, \nu} (\delta_{\alpha\mu} g_{\alpha\nu} + \delta_{\alpha\gamma} g_{\alpha\mu}) (\delta_{\mu\nu} g_{\mu\beta} + \delta_{\mu\beta} g_{\mu\nu}) (\delta_{\alpha\mu} g_{\alpha\nu} + \delta_{\alpha\gamma} g_{\alpha\mu}) g_\beta g_\gamma \\
 &\times \int_{q_1, q_2} \left( \frac{1}{(2\kappa_1)(\kappa_1 + \kappa_2 + \kappa_3)(2\kappa_1 + 2\kappa_2)(2\kappa_1)} \right. \\
 &\left. + \frac{1}{(2\kappa_1)(\kappa_1 + \kappa_2 + \kappa_3)(2\kappa_1 + 2\kappa_2)(2\kappa_2)} \right) + (\beta \leftrightarrow \gamma) \\
 &= -\frac{\lambda g^2}{32} G_\varepsilon^2 \tau^{-\varepsilon} (\delta_{\alpha\beta} g_{\alpha\gamma} + \delta_{\alpha\gamma} g_{\alpha\beta}) (g(g_{\beta\gamma} + g_{\gamma\beta}) + g_{\alpha\beta} g_{\alpha\gamma} + g_{\beta\gamma} g_{\gamma\beta}) I_{12,1} \quad (C1)
 \end{aligned}$$

Thus we obtain the contribution of the diagram Fig. 9(i) to the vertex function  $\Gamma_{\alpha, \alpha\beta}$ :

$$\Gamma_{\alpha, \alpha\beta}^{5(i)} = \frac{\lambda g^2}{16} G_\varepsilon^2 \tau^{-\varepsilon} g_{\alpha\beta} (g(g_{\beta\alpha} + g_{\alpha\beta}) + g_{\alpha\beta} g + g_{\beta\alpha} g_{\alpha\beta}) I_{12,1} \quad (C2)$$

In the same manner we calculate the other diagrams of Fig. 9 and find

$$\Gamma_{\alpha, \alpha\beta}^{5(a)} = \frac{\lambda g^2}{32} G_\varepsilon^2 \tau^{-\varepsilon} g_{\alpha\beta} g (2g + g_{\alpha\beta} + g_{\beta\alpha}) I_{30,1} \quad (C3)$$

$$\Gamma_{\alpha, \alpha\beta}^{5(b)} = \frac{\lambda g^2}{16} G_\varepsilon^2 \tau^{-\varepsilon} g_{\alpha\beta} g (2g + g_{\alpha\beta} + g_{\beta\alpha}) (I_{20,2} + I_{30,1}) \quad (C4)$$

$$\Gamma_{\alpha, \alpha\beta}^{5(c)} = \frac{\lambda g^2}{32} G_\varepsilon^2 \tau^{-\varepsilon} g_{\alpha\beta} g (2g + g_{\alpha\beta} + g_{\beta\alpha}) I_{30,1} \quad (C5)$$

$$\Gamma_{\alpha, \alpha\beta}^{5(d)} = \frac{\lambda g^2}{16} G_\varepsilon^2 \tau^{-\varepsilon} g_{\alpha\beta} g (2g + g_{\alpha\beta} + g_{\beta\alpha}) (2I_{11,2} + I_{12,1}) \quad (C6)$$

$$\Gamma_{\alpha, \alpha\beta}^{5(e)} = \frac{\lambda g^2}{16} G_\varepsilon^2 \tau^{-\varepsilon} g_{\alpha\beta} g (2g + g_{\alpha\beta} + g_{\beta\alpha}) I_{12,1} \quad (C7)$$

$$\Gamma_{\alpha, \alpha\beta}^{5(f)} = \frac{\lambda g^2}{32} G_\varepsilon^2 \tau^{-\varepsilon} g_{\alpha\beta} ((g_{\alpha\beta}^2 + g_{\beta\alpha}^2) + 2gg_{\beta\alpha} + 4g^2) (2I_{11,2} + I_{12,1}) \quad (C8)$$

$$\Gamma_{\alpha, \alpha\beta}^{5(g)} = \frac{\lambda g^2}{16} G_\varepsilon^2 \tau^{-\varepsilon} g_{\alpha\beta} (g(2g + g_{\alpha\beta}) + g_{\alpha\beta} g_{\beta\alpha}) I_{12,1} \quad (C9)$$

$$\Gamma_{\alpha, \alpha\beta}^{5(h)} = \frac{\lambda g^2}{16} G_\varepsilon^2 \tau^{-\varepsilon} g_{\alpha\beta} (g_{\alpha\beta}^2 + g_{\beta\alpha}^2 + 2gg_{\beta\alpha} + 4g^2) I_{20,2} \quad (C10)$$

$$\Gamma_{\alpha, \alpha\beta}^{5(i)} = \frac{3\lambda g^2}{16} G_\varepsilon^2 \tau^{-\varepsilon} g_{\alpha\beta}^2 (g + g_{\beta\alpha}) I_{11,2} \quad (C11)$$

$$\Gamma_{\alpha, \alpha\beta}^{5(k)} = \frac{\lambda g^2}{16} G_\varepsilon^2 \tau^{-\varepsilon} g_{\alpha\beta} (g_{\alpha\beta}^2 + g_{\beta\alpha}^2 + 2gg_{\beta\alpha} + 4g^2) I_{11,2} \quad (C12)$$

Adding up all the two-loop contributions  $\Gamma_{\alpha, \alpha\beta}^{5(a)}, \dots, \Gamma_{\alpha, \alpha\beta}^{5(k)}$  and extracting the singular parts Eq. (A4), we finally get

$$\begin{aligned} \Gamma_{\alpha, \alpha\beta}^{(2-loop)} = & \frac{\lambda g^2 G_\varepsilon^2}{16\varepsilon} \tau^{-\varepsilon} g_{\alpha\beta} \left( \left( \frac{8}{\varepsilon} + 1 \right) g^2 + \left( \frac{4}{\varepsilon} + \frac{1}{2} \right) gg_{\beta\alpha} + \left( \frac{2}{\varepsilon} + 1 \right) g_{\alpha\beta} g_{\beta\alpha} \right. \\ & \left. + \left( \frac{4}{\varepsilon} + \frac{3}{2} - 3 \ln \frac{4}{3} \right) gg_{\alpha\beta} + \left( \frac{1}{\varepsilon} + \frac{3}{2} \ln \frac{4}{3} \right) (g_{\alpha\beta}^2 + g_{\beta\alpha}^2) \right) + O(\varepsilon^0) \end{aligned} \quad (C13)$$

## APPENDIX D: EQUIVALENCE OF THE BOSONIC AND THE LANGEVIN APPROACH

Here we shortly show that a so-called exact microscopic approach to a field theory of DP based on a master equation for bosonic particles leads to the same field theory as the mesoscopic Langevin formulation. For brevity we restrict ourselves to the case of a single species. The generalization to several species is then straightforward.

The microscopic approach is based on the introduction of a Fock space spanned by bosonic multiparticle states  $|\{n\}\rangle$ , where the numbers  $n_i = 0, 1, \dots$  describe the occupation of the lattice sites  $i$  by particles.<sup>(21)</sup> A probability distribution  $P(\{n\})$  translates into a state  $|P\rangle = \sum_{\{n\}} P(\{n\}) |\{n\}\rangle$  and the expectation value of an observable  $A(\{n\})$  is given by

$$\langle A \rangle = \langle \cdot | A(\{a^+ a\}) | P \rangle \quad (D1)$$

Here  $a_i^+$  and  $a_i$  are creation and annihilation operators of particles at the site  $i$ , respectively, and  $\langle \cdot | = \langle 0 | \exp(\sum_i a_i)$  denotes the so-called projection state. In the Fock space formulation the master equation for the temporal evolution of the probability distribution  $P(\{n\}, t)$  takes the form

$$\frac{\partial}{\partial t} |P(t)\rangle = -H |P(t)\rangle \quad (D2)$$

where the Hamiltonian  $H$  describes the different diffusion and reaction processes. In particular the local reactions  $X \rightarrow 0$ ,  $X \rightarrow 2X$ , and  $2X \rightarrow X$  lead to a reaction part  $H_{\text{react}} = \sum_i [\alpha(a_i^+ - 1) a_i + \beta(1 - a_i^+) a_i^+ a_i + \gamma(a_i^+ - 1) a_i^+ a_i^2]$ , where  $\alpha$ ,  $\beta$ , and  $\gamma$  are rates of death, birth and competition of the particles.

Following standard procedures<sup>(22)</sup> one writes the expectation value  $\langle A \rangle$  as a coherent-state path integral

$$\langle A(t) \rangle = \int \mathcal{D}(\hat{\psi}, \psi) A(\{\psi(t)\}) \exp(-S[\hat{\psi}, \psi]) \quad (\text{D3})$$

where  $\psi^+ = 1 + \hat{\psi}$ , and the Lagrangian of the action  $S = \int dt \sum_i \mathcal{L}(\hat{\psi}(\mathbf{x}_i, t), \psi(\mathbf{x}_i, t)) + \text{initial terms}$  is given by

$$\mathcal{L} = \hat{\psi} \partial_t \psi + \lambda \nabla \hat{\psi} \nabla \psi + \mathcal{L}_{\text{react}} \quad (\text{D4})$$

$$\begin{aligned} \mathcal{L}_{\text{react}} &= \alpha \hat{\psi} \psi - \beta \hat{\psi} (1 + \hat{\psi}) \psi + \gamma \hat{\psi} (1 + \hat{\psi}) \psi^2 \\ &= (\alpha - \beta) \hat{\psi} \psi + \hat{\psi} (\gamma \psi - \beta \hat{\psi}) \psi + \gamma (\hat{\psi} \psi)^2 \end{aligned} \quad (\text{D5})$$

Here  $\nabla$  denotes a lattice gradient. For DP  $\alpha$ ,  $\beta$ , and  $\gamma$  are all finite positive parameters with  $\alpha \approx \beta$  at criticality. Thus, the quartic term  $\gamma(\hat{\psi}\psi)^2$  is irrelevant in the continuum limit, and the action takes on the same structure as the dynamic functional Eq. (5). Note the different sign of the ‘‘noise’’ terms  $-\beta\hat{\psi}^2\psi$  and  $+\gamma\hat{\psi}^2\psi^2$ . In case of vanishing death rate  $\alpha$  the birth rate  $\beta$  itself goes to zero at criticality and one has to retain the quartic noise term. Its positive sign corresponds to an ‘‘imaginary’’ anticorrelating Langevin force.

Using the quasicanonical transformation to density variables  $\psi^+ = 1 + \hat{\psi} = \exp(\tilde{n})$  and  $\psi = n \exp(-\tilde{n})$ , we find

$$\begin{aligned} \mathcal{L}_{\text{react}} &= \alpha(1 - \exp(-\tilde{n})) n + \beta(1 - \exp(\tilde{n})) n + \gamma(1 - \exp(-\tilde{n})) n^2 \\ &= (\alpha - \beta) \tilde{n} n + \tilde{n} \left( \gamma n - \frac{\alpha + \beta}{2} \tilde{n} \right) n + \dots \end{aligned} \quad (\text{D6})$$

Thus  $\mathcal{L}_{\text{react}}$  has the same form irrespective of using of density variables or creation-annihilation variables. This shows the equivalence of both approaches as long as  $\alpha \approx \beta \neq 0$ . If the spontaneous decay rate  $\alpha$  is zero,  $\beta$  vanishes at criticality and one is forced to expand the exponentials to higher order terms. However, a further analysis reveals that then the scaling dimension of  $\tilde{n}$  is zero and the expansion of the exponentials is not allowed. One has to resort to the creation-annihilation variables. A naive expansion up to the quartic term  $-\gamma\tilde{n}^2 n^2$  produces a wrong sign of this noise term. This demonstrates that one can use a density description only

as long as spontaneous decay is non-zero. Furthermore, the transformation of the diffusion term  $\nabla\hat{\psi}\nabla\psi$  to density variables leads to a term proportional to  $n(\nabla\hat{n})^2$  in the action  $\mathcal{L}$  (D4). This term represents diffusion noise and is only relevant if the spontaneous decay rate is zero.

## ACKNOWLEDGMENTS

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